A system of Pellian equations and related two-parametric family of quartic Thue equations

Borka Jadrijević

Abstract

We show that solving of the two-parametric family of quartic Thue equations

$$x^4 - 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 + 2mnxy^3 + y^4 = 1,$$

using the method of Tzanakis, reduces to solving the system of Pellian equations

$$V^2 - \left(m^2 + 2\right)U^2 = -2, \qquad Z^2 - \left(n^2 - 2\right)U^2 = 2,$$

where parameters m and $n \neq 0, \pm 1$ are integers. The main result in this paper can be stated as follows: If $|m|$ and $|n|$ are sufficiently large and have sufficiently large common divisor, then the system has only the trivial solutions $(V, Z, U) = (\pm m, \pm n, \pm 1)$, which implies that the original Thue equation also has only the trivial solutions $(x, y) = (\pm 1, 0), (0, \pm 1)$.

1 Introduction

Let $F \in \mathbb{Z}[X, Y]$ be a homogeneous irreducible polynomial of degree ≥ 3 and $t \neq 0$ a fixed integer. Then Diophantine equation $F(x, y) = t$ is called a Thue equation in the honour of A. Thue, who proved in 1909 [24] that such an equation has only finitely many solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. Thue's proof is not effective. Using estimates for linear forms in logarithms of algebraic numbers, Baker [1] could give effective upper bound for the solutions of Thue equation. Since that time, general powerful methods have been developed for the explicit solution of Thue equations (see [21, 27, 5]), following

[0] *2000 Mathematics Subject Classification.* Primary: 11D59; Secondary: 11D25, 11B37, 11J86.

Key words. Thue equations, simultaneous Pellian equations, linear forms in logarithms

from Baker's work. In 1990, Thomas [23] investigated for the first time a parametrized family of Thue equations. Since then, several families have been studied (see [12] for references). In particular, quartic families have been considered in [6, 10, 12, 14, 16, 20, 22, 25, 28, 29].

In this paper, we consider the equation

$$x^4 - 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 + 2mnxy^3 + y^4 = 1. \tag{1}$$

Using the method of Tzanakis, given in [26], we will show that solving equation (1) reduces to solving the system of Pellian equations

$$V^2 - \left(m^2 + 2\right)U^2 = -2, \tag{2}$$
$$Z^2 - \left(n^2 - 2\right)U^2 = 2, \tag{3}$$

and we prove, roughly speaking, that if $|m|$ and $|n|$ are sufficiently large and have sufficiently large common divisor, then the system has only the trivial solutions $(V, Z, U) = (\pm m, \pm n, \pm 1)$, which implies (1) equation has only the trivial solutions $(\pm 1, 0)$, $(0, \pm 1)$.

We will find a lower bound for solutions of this system using the "congruence method" introduced in [11] by Dujella and Pethő and used also in [8, 9, 10]. The comparison of this lower bound with an upper bound obtained from a theorem of Baker and Wüstholz [3] lead to the main result of this paper.

In [26], Tzanakis considered a certain class of quartic Thue equations whose corresponding quartic field $\mathbf{K}$ is totally real, Galois and non-cyclic. Tzanakis showed that solving the equation, under above assumptions on $\mathbf{K}$, reduces to solving a system of Pellian equations having one common unknown. Such reduction has certain advantages. Dealing with system of Pellian equations, all algebraic - arithmetic data we need, are easily accessible. Furthermore, methods based on the theory of linear forms in logarithms lead to a linear form in three logarithms and two unknown integral coefficients instead, the known direct general methods such as [27], to a linear form in four logarithms and three unknown integral coefficients.

The main result of the present paper is the following theorem.

Theorem 1 *For every* $0.5 < \varepsilon \leq 1$ *there exists an effectively computable constant* $C(\varepsilon)$ *such that if* $m \neq 0$, $\max\{|m|, |n|\} \geq C(\varepsilon)$ *and*

$$\gcd(m, n) \geq \max\{|m|^\varepsilon, |n|^\varepsilon\},$$

then system of Pellian equations (2) *and* (3) *has only the trivial solutions* $(V, Z, U) = (\pm m, \pm n, 1)$. *In particular, we may take* $C(0.999) = 10^{27}$,

$C(0.99) = 10^{30}$, $C(0.9) = 10^{48}$, $C(0.8) = 10^{71}$, $C(0.7) = 10^{116}$, $C(0.6) = 10^{255}$, $C(0.51) = 10^{3138}$, $C(0.501) = 10^{36836}$.

Corollary 1 *For every* $0.5 < \varepsilon \leq 1$ *there exists an effectively computable constant* $C(\varepsilon)$ *such that if* $m \neq 0$, $\max\{|m|, |n|\} \geq C(\varepsilon)$ *and*

$$\gcd(m,n) \geq \max\{|m|^{\varepsilon},\ |n|^{\varepsilon}\},$$

then Thue equation (1) *has only the trivial solutions* $(x, y) = (\pm 1, 0)$, $(0, \pm 1)$.

Remark 1 In [13], it is proven that for all integers m and n there are no non-trivial solutions of (1) satisfying the additional condition $\gcd(xy, mn) = 1$. The result is obtained by considering three cases: $m = n$, $m = 2n$, $n = 2m$. These cases can be completely solved by applying a theorem of Bennett [4] on simultaneous approximations of algebraic numbers. The case $m = 2n$ can be considered as a special case of the Thue equation

$$x^4 - 4cx^3y + (6c + 2)x^2y^2 + 4cxy^3 + y^4 = 1,$$

which was completely solved in [10].

2 Small values of parameters

Proposition 1 *Equation* (1) *has only the trivial solutions* $(x, y) = (\pm 1, 0)$, $(0, \pm 1)$ *in the following cases:*

i) $|n| \leq 1$,

ii) $m = 0$ *and* $2(n^2 - 1)$ *is not a perfect square.*

PROOF. The statement of the proposition is trivially true for $n = 0$. On the other hand, for $n = \pm 1$ we have

$$x^4 \mp 2mx^3y + 2m^2x^2y^2 \pm 2mxy^3 + y^4 = \left(x^2 \mp mxy - y^2\right)^2 + x^2y^2\left(m^2 + 2\right),$$

and therefore the statement is true in this case too.

For $m = 0$ we have

$$x^4 + 2(1 - n^2)x^2y^2 + y^4 = 1, \tag{4}$$

which is special case of equation

$$x^4 - Kx^2y^2 + y^4 = 1, \tag{5}$$

where K is a positive integer. This equation was considered by Cusick in [7] and his result is generalized by Walsh in [30]. Using Ljunggren's results [15], it is proven that equation (5) does not have any solution in positive integers x and y except for the trivial cases where K is a square and $x = 1$ or $y = 1$. In our special case we have: If the sequence (n_k) is defined by

$$n_0 = 1, \quad n_1 = 3, \quad n_{k+2} = 6n_{k+1} - n_k, \quad k \geq 0, \tag{6}$$

then, for $n = n_k$ and $k \geq 1$, all non-trivial solutions of equation (4) are given by

$$(x, y) = \left(\pm 1, \pm\sqrt{2\left(n^2 - 1\right)}\right) \text{ and } \left(\pm\sqrt{2\left(n^2 - 1\right)}, \pm 1\right).$$

For all other values of n we have only the trivial solutions $(x, y) = (\pm 1, 0)$, $(0, \pm 1)$. ■

Remark 2 For $m = 0$ system (2) and (3) has non-trivial positive solution $(V, Z, U) = \left(n\sqrt{2\left(n^2 - 1\right)},\ 2n^2 - 3,\ 2n^2 - 1\right)$ if $n = n_k$, $k \geq 1$, where sequence (n_k) is given by (6).

Let us note that, because of homogeneity and symmetry of equation (1), it is enough to consider the cases when m and n are nonnegative and find only all positive solutions. More precisely, $(x, y) = (a, b)$ is a solution of equation (1) if and only if $(x, y) = (b, a)$ is a solution of equation

$$x^4 + 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 - 2mnxy^3 + y^4 = 1.$$

Further, if $(x, y) = (a, b)$ is solution of equation (1) then $(x, y) = (-a, -b)$, $(b, -a)$, $(-b, a)$ are solutions too. Thus, we will consider equation (1) for $m \geq 1$ and $n \geq 2$.

3 The method of Tzanakis

In this section we will describe the method of Tzanakis [26] for solving quartic Thue equations whose corresponding quartic field $\mathbf{K}$ has the properties stated in Section 1.

Consider the quartic Thue equation

$$f(x, y) = t \tag{7}$$

$$f(x, y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4 \in \mathbf{Z}[x, y], \ \ a_0 > 0,$$

whose corresponding quartic field $\mathbf{K}$ is Galois and non-cyclic. By [18], this condition on $\mathbf{K}$ is equivalent with $\mathbf{K}$ having three quadratic subfields, which happens exactly when the cubic resolvent of the quartic Thue equation has three distinct rational roots.

It is more convenient to consider the cubic equation

$$4\rho^3 - g_2\rho - g_3 = 0 \tag{8}$$

with roots opposite to those of the cubic resolvent of the quartic equation $f(x,1) = 0$. Here g_2 and g_3 are invariants of the form:

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 \in \frac{1}{12}\mathbb{Z}, \qquad g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \in \frac{1}{432}\mathbb{Z}\,.$$

Therefore, the above conditions on $\mathbf{K}$ are equivalent to the fact that the cubic equation (8) has three rational roots ρ_1, ρ_2, ρ_3. Assuming that $\mathbf{K}$ is not totally complex, then by classification of Nagell (Table in page 351 of [18]), $\mathbf{K}$ is totally real, in fact, it is a compositum of two real quadratic fields. This happens exactly when

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}. \tag{9}$$

Let $H(x,y)$ and $G(x,y)$ be the quartic and sextic covariants of $f(x,y)$, respectively (see [17, Chapter 25]), i.e.

$$H(x,y) = -\frac{1}{144}\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} \in \frac{1}{48}\mathbf{Z}[x,y]\,,$$

$$G(x,y) = -\frac{1}{8}\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{vmatrix} \in \frac{1}{96}\mathbf{Z}[x,y]\,.$$

Then

$$4H^3 - g_2Hf^2 - g_3f^3 = G^2. \tag{10}$$

If we put $H = \frac{1}{48}H_0$, $G = \frac{1}{96}G_0$, $\rho_i = \frac{1}{12}r_i$, $i = 1,2,3$, then $H_0, G_0 \in \mathbb{Z}\,[x,y]$, $r_i \in \mathbb{Z}$, $i = 1,2,3$. Since f and H are relatively prime in $\mathbb{Q}\,[x,y]$, then, in a view of (10),

$$(H_0 - 4r_1f)(H_0 - 4r_2f)(H_0 - 4r_3f) = 3G_0^2$$

and the tree factors on the left-hand side are pairwise relatively prime in $\mathbb{Z}[x,y]$. So, there exist positive square-free integers k_1, k_2, k_3 and quadratic forms G_1, G_2, $G_3 \in \mathbb{Z}[x,y]$ such that

$$H_0 - 4r_i f = k_i G_i^2, \quad i = 1,2,3$$

and $k_1k_2k_3(G_1G_2G_3)^2 = 3G_0^2$. If $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ is a solution of (7), then

$$\begin{aligned} k_2G_2^2 - k_1G_1^2 &= 4(r_1 - r_2)t, & (11)\\ k_3G_3^2 - k_1G_1^2 &= 4(r_1 - r_3)t. & (12) \end{aligned}$$

In this way, solving the Thue equation (7) reduces to solving the system of Pellian equations (11) and (12) with one common unknown.

4 Binary recursive sequences

Let us apply the method from Section 3 to the equation

$$f(x,y) = x^4 - 2mnx^3y + 2\left(m^2 - n^2 + 1\right)x^2y^2 + 2mnxy^3 + y^4 = 1\,.$$

We have

$$\begin{aligned} g_2 &= 1 + m^2n^2 + \frac{1}{3}\left(m^2 - n^2 + 1\right)^2,\\ g_3 &= -\frac{1}{54}\left(m^2 + 2n^2 - 2\right)\left(2m^2 + n^2 + 2\right)\left(m^2 - n^2 + 4\right), \end{aligned}$$

$$\rho_1 = \frac{1}{6}m^2 - \frac{1}{6}n^2 + \frac{2}{3}, \quad \rho_2 = \frac{1}{6}m^2 + \frac{1}{3}n^2 - \frac{1}{3}, \quad \rho_3 = -\frac{1}{3}m^2 - \frac{1}{6}n^2 - \frac{1}{3}.$$

If $|n| \geq 2$, then

$$\begin{aligned} \frac{a_1^2}{a_0} - a_2 - \rho_1 &= \frac{1}{4}\left(m^2 + 2\right)\left(n^2 - 2\right) \geq 0,\\ \frac{a_1^2}{a_0} - a_2 - \rho_2 &= \frac{1}{4}m^2\left(n^2 - 2\right) \geq 0,\\ \frac{a_1^2}{a_0} - a_2 - \rho_3 &= \frac{1}{4}n^2\left(m^2 + 2\right) \geq 0. \end{aligned}$$

Thus, condition (9) is clearly satisfied for $|n| \geq 2$.

Furthermore, we obtain

$$\begin{aligned} H_0 - 4r_1f &= 12\left(n^2 - 2\right)\left(m^2 + 2\right)\left(x^2 + y^2\right)^2,\\ H_0 - 4r_2f &= 12\left(n^2 - 2\right)\left(mx^2 + 2nxy - my^2\right)^2,\\ H_0 - 4r_3f &= 12\left(m^2 + 2\right)\left(-nx^2 + 2mxy + ny^2\right)^2. \end{aligned}$$

By putting $k_1 = 3\left(n^2-2\right)\left(m^2+2\right)$, $k_2 = 3\left(n^2-2\right)$, $k_3 = 3\left(m^2+2\right)$,

$$\begin{aligned} U &= \frac{G_1}{2} = x^2+y^2, \\ V &= \frac{G_2}{2} = mx^2+2nxy-my^2, \\ Z &= \frac{G_3}{2} = -nx^2+2mxy+ny^2. \end{aligned}$$

we get, from (11) and (12), the system of Pellian equations

$$V^2-\left(m^2+2\right)U^2 = -2, \tag{13}$$
$$Z^2-\left(n^2-2\right)U^2 = 2. \tag{14}$$

Consider the system of Pellian equations (13) and (14). Neither m^2+2 nor n^2-2 is a square and both $\mathbb{Q}\left(\sqrt{m^2+2}\right)$ and $\mathbb{Q}\left(\sqrt{n^2-2}\right)$ are real quadratic real number fields. Moreover, by [19, Theorem 105], $m^2+1+m\sqrt{m^2+2}$ and $n^2-1+n\sqrt{n^2-2}$ are non-trivial units of norm 1 in number rings $\mathbb{Z}\left[\sqrt{m^2+2}\right]$ and $\mathbb{Z}\left[\sqrt{n^2-2}\right]$, respectively. By [19, Theorem 108a], we find that

$$\begin{aligned} v_1+u_1\sqrt{m^2+2} &= m+\sqrt{m^2+2}, \\ v_2+u_2\sqrt{m^2+2} &= -m+\sqrt{m^2+2} \end{aligned}$$

are the possible fundamental solutions of equation (13). By [19, Chapter VI, 58.], these solutions belong to the same class, so we have only one fundamental solution $m+\sqrt{m^2+2}$.

Similarly, by [19, Theorem 108], we find that $n+\sqrt{n^2-2}$ is the only fundamental solution of equation (14).

Hence, all solutions of equation (13) in positive integers are given by

$$v+u\sqrt{m^2+2} = \left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)^k, \tag{15}$$

where $k \in \mathbb{Z}$ and $k \geq 0$ or by $u = U_k$ and $v = V_k$, where the sequences (U_k) and (V_k) are defined by the recurrences

$$U_0 = 1, \quad U_1 = 2m^2+1, \quad U_{k+2} = 2\left(m^2+1\right)U_{k+1}-U_k, \ k \geq 0; \tag{16}$$

$$V_0 = m, \quad V_1 = m\left(2m^2+3\right), \quad V_{k+2} = 2\left(m^2+1\right)V_{k+1}-V_k, \ k \geq 0.$$

All solutions of equation (14) in positive integers are given by

$$z+t\sqrt{n^2-2} = \left(n+\sqrt{n^2-2}\right)\left(n^2-1+n\sqrt{n^2-2}\right)^l \tag{17}$$

where $l \in \mathbb{Z}$ and $l \geq 0$ or by $t = T_l$ and $z = Z_l$, where the sequences (T_l) and (Z_l) are defined by the recurrences

$$T_0 = 1, \quad T_1 = 2n^2 - 1, \quad T_{l+2} = 2\left(n^2 - 1\right) T_{l+1} - T_l, \ l \geq 0; \tag{18}$$

$$Z_0 = n, \quad Z_1 = n\left(2n^2 - 3\right), \quad Z_{l+2} = 2\left(n^2 - 1\right) Z_{l+1} - Z_l, \ l \geq 0.$$

In this way we reformulated the system of Pell's equations (13) and (14) to the Diophantine equation of the form

$$U_k = T_l$$

in integers $k, l \geq 0$. In order to prove Theorem 1, it suffices to show that $U_k = T_l$ implies $k = l = 0$ in all these special cases.

Solving recurrences (16) and (18) we find

$$\begin{aligned} U_k &= \frac{1}{2\sqrt{m^2+2}}\Big[\left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)^k \\ &- \left(m-\sqrt{m^2+2}\right)\left(m^2+1-m\sqrt{m^2+2}\right)^k\Big], \end{aligned} \tag{19}$$

$$\begin{aligned} T_l &= \frac{1}{2\sqrt{n^2-2}}\Big[\left(n+\sqrt{n^2-2}\right)\left(n^2-1+n\sqrt{n^2-2}\right)^l \\ &- \left(n-\sqrt{n^2-2}\right)\left(n^2-1-n\sqrt{n^2-2}\right)^k\Big]. \end{aligned} \tag{20}$$

5 Linear form in logarithms of algebraic numbers

Lemma 1 *If $m \geq n \geq 2$, $U_k = T_l$ and $k \neq 0$, then the following hold:*

1^0

$$0 < l \log\left(n^2 - 1 + n\sqrt{n^2 - 2}\right) - k \log\left(m^2 + 1 + m\sqrt{m^2 + 2}\right) +$$

$$+ \log \frac{\sqrt{m^2+2}\left(n + \sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m + \sqrt{m^2+2}\right)} < 0.406\left(m^2 + 1 + m\sqrt{m^2+2}\right)^{-2k}.$$

2^0 $k < l < 0.5674 \cdot t(m) \cdot k$, *where* $t(m) = \log\left(m^2 + 1 + m\sqrt{m^2 + 2}\right)$.

PROOF. $\mathbf{1}^0$ If we put

$$\begin{aligned} P &= \left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)^k, \\ Q &= \left(n+\sqrt{n^2-2}\right)\left(n^2-1+n\sqrt{n^2-2}\right)^l, \end{aligned}$$

then

$$\begin{aligned} P^{-1} &= \frac{-1}{2}\left(m-\sqrt{m^2+2}\right)\left(m^2+1-m\sqrt{m^2+2}\right)^k, \\ Q^{-1} &= \frac{1}{2}\left(n-\sqrt{n^2-2}\right)\left(n^2-1-n\sqrt{n^2-2}\right)^l. \end{aligned}$$

Now, from (19) and (20) it follows that the relation $U_k = T_l$ implies

$$P + 2P^{-1} = A\left(Q - 2Q^{-1}\right),$$

where

$$A = \sqrt{\frac{m^2+2}{n^2-2}} > 1.$$

It is clear that $P > 1$, $Q > 1$, and from

$$AQ - P = 2\left(AQ^{-1} + P^{-1}\right) > \frac{1}{A}Q^{-1} - P^{-1} = \frac{1}{A}\left(P - AQ\right)Q^{-1}P^{-1},$$

it follows that $AQ > P$, which implies $Q^{-1} < AP^{-1}$. Thus, we have

$$AQ - P = 2AQ^{-1} + 2P^{-1} < 2A^2P^{-1} + 2P^{-1} = 2\left(A^2+1\right)P^{-1}.$$

Since $k \geq 1$, we have

$$2\left(A^2+1\right)P^{-2} \leq \frac{2\left(\frac{m^2+2}{n^2-2}+1\right)}{\left(\left(m+\sqrt{m^2+2}\right)\left(m^2+1+m\sqrt{m^2+2}\right)\right)^2} = f(m,n).$$

For $m \geq n \geq 2$, function $f(m,n)$ is decreasing in the both variables, which implies

$$2\left(A^2+1\right)P^{-2} \leq f(2,2) < 4.2 \times 10^{-3}.$$

Thus, we have

$$\frac{AQ-P}{AQ} < \frac{2\left(A^2+1\right)}{A}P^{-1}Q^{-1} < 2\left(A^2+1\right)P^{-2} < 4.2 \times 10^{-3}.$$

Hence,

$$0 < \log\frac{AQ}{P} = -\log\left(1-\frac{AQ-P}{AQ}\right) < \left(\frac{AQ-P}{AQ}\right) + \left(\frac{AQ-P}{AQ}\right)^2$$
$$< \left[1+2\left(A^2+1\right)P^{-2}\right]\cdot 2\left(A^2+1\right)P^{-2} <$$
$$< \left(1+4.2\times 10^{-3}\right)\cdot\frac{2\left(\frac{m^2+2}{n^2-2}+1\right)}{\left(m+\sqrt{m^2+2}\right)^2}\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k}.$$

The function

$$f_1(m,n) = \frac{2\left(\frac{m^2+2}{n^2-2}+1\right)}{\left(m+\sqrt{m^2+2}\right)^2}$$

is decreasing in both variables, so $f_1(m,n) \le f_1(2,2) < 0.4041$. Thus we have

$$0 < \log\frac{AQ}{P} < 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k},$$

which implies the assertion.

$\mathbf{2}^0$ From relations (19) and (20) it is clear that $k < l$. We have, by $\mathbf{1}^0$, that

$$l\log\left(n^2-1+n\sqrt{n^2-2}\right) - k\log\left(m^2+1+m\sqrt{m^2+2}\right)$$
$$< \quad 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k} - \log\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)} \le$$
$$\le \quad 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2} - \log\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)} = g\left(m,n\right).$$

Since $g\left(m,n\right) = g_1\left(m\right) + g_2\left(n\right)$, where

$$g_1\left(m\right) = 0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2} + \log\frac{m+\sqrt{m^2+2}}{\sqrt{m^2+2}},$$
$$g_2\left(n\right) = \log\frac{\sqrt{n^2-2}}{n+\sqrt{n^2-2}},$$

then $g\left(m,n\right) \le g_1\left(m\right) + g_2\left(m\right) = g_3\left(m\right)$. Function $g_3\left(m\right)$ is continuous, strictly increasing and $\lim\limits_{m\to\infty} g_3\left(m\right) = 0$ and we conclude that $g_3\left(m\right) < 0$ for

$m \geq 2$. Thus, we have

$$l \log\left(n^2 - 1 + n\sqrt{n^2-2}\right) - k \log\left(m^2 + 1 + m\sqrt{m^2+2}\right) < 0,$$

which implies

$$\frac{l}{k} < \frac{\log\left(m^2 + 1 + m\sqrt{m^2+2}\right)}{\log\left(n^2 - 1 + n\sqrt{n^2-2}\right)} = T\left(m,n\right).$$

Since function $T\left(m,n\right)$ attains its greatest value at $n = 2$ for every m, we have

$$\frac{l}{k} < T\left(m,2\right) < 0.5674 \cdot \log\left(m^2 + 1 + m\sqrt{m^2+2}\right),$$

which implies the assertion. ■

Lemma 2 *If* $1 \leq m < n$, $n \geq 3$, $U_k = T_l$ *and* $l \neq 0$, *then the following hold:*

1^0

$$0 < l \log\left(n^2 - 1 + n\sqrt{n^2-2}\right) - k \log\left(m^2 + 1 + m\sqrt{m^2+2}\right)$$
$$-\log\frac{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)} < 0.211\left(n^2 - 1 + n\sqrt{n^2-2}\right)^{-2l},$$

2^0 $l < k < 0.76 \cdot t\left(n\right) \cdot l$, *where* $t\left(n\right) = \log\left(1.354\left(n^2 - 1 + n\sqrt{n^2-2}\right)\right)$.

Proof. Similarly as in the proof of Lemma 2. For details see [13, Lemma 5.14]. ■

Note that the case $m = 1$ and $n = 2$ we have to consider separately. Here system (13) and (14) has the form

$$\begin{aligned} V^2 - 3U^2 &= -2, \\ Z^2 - 2U^2 &= 2. \end{aligned}$$

We can show, by using a slight modification of [11, Lemma 5 a)], which is a variant of the Baker-Davenport reduction procedure [2], that this system has two positive solutions $\left(V,Z,U\right) = \left(1,2,1\right)$, $\left(41,71,58\right)$, so corresponding Thue equation has solutions given by $\left(x,y\right) = \left(\pm1,0\right)$, $\left(0,\pm1\right)$, $\left(4,5\right)$, $\left(-4,-5\right)$, $\left(5,-4\right)$, $\left(-5,4\right)$. For details see [13, Lemma 5.10].

6 The congruence method

Let $g = \gcd(m, n)$.

Lemma 3 *Let the sequences (U_k) and (T_l) be defined by (16) and (18). Then for all $k, l \geq 0$ we have*

$$U_k \equiv \left[k(k+1)m^2 + 1\right] \pmod{2g^4}, \tag{21}$$
$$T_l \equiv (-1)^{l+1}\left[l(l+1)n^2 - 1\right] \pmod{2g^4}. \tag{22}$$

Proof. Both relations are obviously true for $k, l \in \{0, 1\}$.
Assume that (21) is valid for $k-1$ and k. Then

$$\begin{aligned}
U_{k+1} &= 2\left(m^2+1\right)U_k - U_{k-1} \\
&\equiv 2\left(m^2+1\right)\left(k^2m^2+km^2+1\right) - \left((k-1)^2m^2+(k-1)m^2+1\right) \\
&= \left(2k(k+1)m^4 + \left(k^2+3k+2\right)m^2+1\right) \\
&\equiv \left[(k+1)((k+1)+1)m^2+1\right] \pmod{2g^4}.
\end{aligned}$$

Assume that (22) is valid for $l-1$ and l. Then

$$\begin{aligned}
T_l &= 2\left(n^2-1\right)T_l - T_{l-1} \\
&\equiv 2\left(n^2-1\right)(-1)^{l+1}\left(l^2n^2+ln^2-1\right) - (-1)^l\left((l-1)^2n^2+(l-1)n^2-1\right) \\
&= 2(-1)^{l+1}\left(l^2+l\right)n^4 + (-1)^l\left(l^2+3l+2\right)n^2 - (-1)^l \\
&\equiv (-1)^{l+2}\left[(l+1)((l+1)+1)n^2-1\right] \pmod{2g^4}.
\end{aligned}$$

∎

Suppose that k and l are positive integers such that $U_k = T_l$. Then, of course, $U_k \equiv T_l \pmod{2g^4}$. Furthermore suppose that $m = m_1 g$, $n = n_1 g$ and $g > 1$. By Lemma 3, we have $1 \equiv (-1)^l \pmod{2g^2}$ and therefore l is even, say $l = 2l_1$. Further, Lemma 3 implies

$$k(k+1)m^2 + 1 \equiv 1 - 2l_1(2l_1+1)n^2 \pmod{2g^4},$$

and

$$k(k+1)m_1^2 \equiv -2l_1(2l_1+1)n_1^2 \pmod{2g^2},$$

which implies

$$\frac{k(k+1)}{2}m_1^2 \equiv -l_1(2l_1+1)n_1^2 = -\frac{l(l+1)}{2}n_1^2 \pmod{2g^2}. \tag{23}$$

Consider the positive integer

$$Y=\frac{k\,(k+1)}{2}m_1^2+\frac{l\,(l+1)}{2}n_1^2.$$

1) Let $m\geq n$. Assume that $l\leq\frac{g}{m_1}$. We have by Lemma 1, $\mathbf{2^0}$, that $k<l$, and since $n_1\leq m_1$ we also have

$$\frac{l\,(l+1)}{2}n_1^2\leq\frac{l\,(l+1)}{2}m_1^2\leq g^2$$

and

$$\frac{k\,(k+1)}{2}m_1^2<\frac{l\,(l+1)}{2}m_1^2\leq g^2.$$

So, we have $0<Y<2g^2$ and, by (23), $Y\equiv 0 \pmod{2g^2}$, a contradiction.

2) Let $m<n$. Assume that $k\leq\frac{g}{n_1}$. We have, by Lemma 2, $\mathbf{2^0}$, that $l<k$, and since $m_1<n_1$ we also have

$$\frac{k\,(k+1)}{2}m_1^2<\frac{k\,(k+1)}{2}n_1^2\leq g^2$$

and

$$\frac{l\,(l+1)}{2}n_1^2<\frac{k\,(k+1)}{2}n_1^2\leq g^2.$$

So, we have $0<Y<2g^2$, and by (23), we obtain a contradiction as before.

Therefore we proved

Proposition 2 *If $U_k=T_l$, $(k,l)\neq(0,0)$, $g=\gcd(m,n)>1$, $m=m_1g$, $n=n_1g$, then we have:*

1) if $m\geq n$ then $l>\frac{g}{m_1}$,

2) if $m<n$ then $k>\frac{g}{n_1}$.

Corollary 2 *If $U_k=T_l$, $(k,l)\neq(0,0)$, $g=\gcd(m,n)>1$, $m=m_1g$, $n=n_1g$, then we have:*

1) if $m\geq n$ then $U>\left(2n^2-3\right)^{\frac{g}{m_1}}$,

2) if $m<n$ then $U>\left(1.8m^2+0.9\right)^{\frac{g}{n_1}}$.

PROOF. From relations (20) and (19) we have

$$U > \left(n^2 - 1 + n\sqrt{n^2-2}\right)^l > \left(2n^2 - 3\right)^l,$$

$$U > \frac{m+\sqrt{m^2+2}}{2\sqrt{m^2+2}}\left(m^2+1+m\sqrt{m^2+2}\right)^k > \frac{9}{10}\left(2m^2+1\right)^k > \left(1.8m^2+0.9\right)^k,$$

respectively, so assertions **1)** and **2)** follow directly from Proposition 2. ■

7 The proof of Theorem 1

Now we will apply the following famous theorem of Baker and Wüstholz [3]:

Theorem 2 *For a linear form $\Lambda \neq 0$ in logarithms of s algebraic numbers $\alpha_1, \dots, \alpha_s$ with rational integer coefficients $b_1, \dots, b_s$ we have*

$$\log \Lambda \geq -18(s+1)!\, s^{s+1}(32d)^{s+2} h'(\alpha_1)\cdots h'(\alpha_s)\log(2sd)\log G\,,$$

where $G = \max\{|b_1|, \dots, |b_s|\}$, and where d is the degree of the number field generated by $\alpha_1, \dots, \alpha_s$.

Here

$$h'(\alpha) = \frac{1}{d}\max\left\{h(\alpha), |\log\ \alpha|, 1\right\},$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α.

1) Let $m \geq n$, $g = \gcd(m,n)$, $n = n_1 g$, $m = m_1 g = Bn_1 g$. We will apply Theorem 2 to the form from Lemma 1. We have: $s = 3$, $d = 4$, $G = l$,

$$\alpha_1 = n^2 - 1 + n\sqrt{n^2-2}, \qquad \alpha_2 = m^2 + 1 + m\sqrt{m^2+2},$$

$$\alpha_3 = \frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}.$$

Under the assumption $g > 1$, we have

$$h'(\alpha_1) = \frac{1}{2}\log\left(n^2-1+n\sqrt{n^2-2}\right) < \frac{1}{2}\log\left(2n^2\right) =$$

$$= \log\left(\sqrt{2}n_1 g\right) < 4.329 \cdot \log\left(\sqrt{2}n_1\right)\cdot \log g$$

$$h'(\alpha_2)=\frac{1}{2}\log\left(m^2+1+m\sqrt{m^2+2}\right)<\frac{1}{2}\log\left(\left(2+\sqrt{2}\right)m^2\right)$$
$$<\log(2Bn_1g)<2.886\cdot\log(2Bn_1)\cdot\log g$$

Furthermore, we find that

$$\alpha_3=|\alpha_3|=\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}<\frac{n\sqrt{m^2+2}}{m\sqrt{n^2-2}},$$

and the conjugates of α_3 satisfy

$$\alpha_3'=\left|\alpha_3'\right|=\frac{\sqrt{m^2+2}\left(n-\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}<1,$$

$$\left|\alpha_3''\right|=\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(\sqrt{m^2+2}-m\right)}<\frac{2n\left(m^2+2\right)}{\sqrt{n^2-2}},$$

$$\left|\alpha_3'''\right|=\frac{\sqrt{m^2+2}\left(n-\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(\sqrt{m^2+2}-m\right)}<\frac{m^2+2}{n^2-2}.$$

Therefore,

$$h'(\alpha_3)=h\left(\frac{\sqrt{m^2+2}\left(n+\sqrt{n^2-2}\right)}{\sqrt{n^2-2}\left(m+\sqrt{m^2+2}\right)}\right)=\frac{1}{4}\log\left[\left(n^2-2\right)^2\prod_{i=1}^{4}\max_j\left(1,\left|\alpha^{(j)}\right|\right)\right]$$
$$<\frac{1}{4}\log\left(\frac{2n^2}{m}\left(m^2+2\right)^{\frac{5}{2}}\right)<7.576\cdot\log\left(\sqrt{2}n_1B^{\frac{5}{7}}\right)\cdot\log g.$$

Hence, Theorem 2 implies

$$\log\Lambda>-3.617\times10^{17}\cdot\log\left(\sqrt{2}n_1\right)\cdot\log(2Bn_1)\cdot\log\left(\sqrt{2}n_1B^{\frac{5}{7}}\right)\cdot(\log g)^3\cdot\log l. \tag{24}$$

On the other hand, Lemma 1 implies

$$\begin{aligned}\log\Lambda &< \log\left[0.406\left(m^2+1+m\sqrt{m^2+2}\right)^{-2k}\right]<\\ &< -2k\log\left(m^2+1+m\sqrt{m^2+2}\right)<-2k\cdot\log g^2< \qquad (25)\\ &< 7.04\cdot\frac{l}{t(m)}\cdot\log g,\end{aligned}$$

where

$$t(m) = \log\left(m^2 + 1 + m\sqrt{m^2+2}\right) < 2\log(2Bn_1 g)$$
$$< 5.773 \cdot \log(2Bn_1) \cdot \log g \tag{26}$$

Combining (24), (25) and (26) we obtain

$$\frac{l}{\log l} < F(B, n_1) \cdot \log^3 g,$$

where

$$\begin{aligned} F(B, n_1) &= 2.967 \times 10^{17} \cdot \log\left(\sqrt{2}n_1\right) \cdot \log^2(2Bn_1) \cdot \log\left(\sqrt{2}n_1 B^{\frac{5}{7}}\right) < \\ &< 2.967 \times 10^{17} \cdot \log^4(2Bn_1) = 2.967 \times 10^{17} \cdot \log^4(2m_1). \end{aligned}$$

By Proposition 2, **1)**, we have $\frac{g}{m_1} < l$, which implies

$$\begin{aligned} \frac{l}{\log l} &< 2.967 \times 10^{17} \cdot \log^4(2m_1) \cdot \log^3(lm_1) < \\ &< 7.133 \times 10^{18} \cdot \log^7(2m_1) \cdot \log^3 l. \end{aligned} \tag{27}$$

If (27) implies $l < l_0$, then

$$g < l \cdot m_1 < l_0 \cdot m_1.$$

For every $m_1 \geq 1$, (27) has form

$$l < K \cdot \log^4 l, \tag{28}$$

where $K = K(m_1) = 7.133 \times 10^{18} \cdot \log^7(2m_1) > 5.483 \times 10^{17}$. We have

$$K < K \log^4 K,$$

which implies $K < l_0$. Therefore, we can assume that $l_0 = K^{1+t_0}$, where $t_0 > 0$. Let us define a function $t_{\min}(K)$, which is implicitly given by

$$K \log^4 K^{1+t_{\min}} = K^{1+t_{\min}},$$

with $t_{\min}(K) > 0$. We find that $t_{\min}(K)$ is a continuous and decreasing function, so we have $t_{\min}(K) < t_{\min}\left(5.483 \times 10^{17}\right) < 0.396$. Hence, we may take $t_0 = \frac{3}{7}$, i.e.

$$l_0 = K^{\frac{10}{7}} < 8.576 \times 10^{26} \log^{10}(2m_1),$$

which implies

$$g < K^{\frac{10}{7}} m_1 < 8.576 \times 10^{26} \cdot m_1 \cdot \log^{10}(2m_1). \tag{29}$$

Assume that $g = m^{\varepsilon}$, $0 < \varepsilon \le 1$. Then, from (29), we have

$$m^{\varepsilon} < G \cdot m^{(1-\varepsilon)} \cdot \log^{10}\left(2m^{(1-\varepsilon)}\right), \tag{30}$$

where $G = 8.576 \times 10^{26}$. Let us define a function $M_0(\varepsilon)$ which is implicitly given by

$$M_0^{\varepsilon} = G \cdot M_0^{1-\varepsilon} \cdot \log^{10}\left(2M_0^{(1-\varepsilon)}\right), \tag{31}$$

with $M_0(\varepsilon) \ge 2$. We find that $M_0(\varepsilon)$ is defined for every $0.5 < \varepsilon \le 1$ and $M_0(\varepsilon)$ is a strictly decreasing function of ε. Hence $M_0(\varepsilon) \ge M_0(1) > 2.195 \times 10^{25}$.

If $0 < \varepsilon \le 0.5$ then (30) is satisfied for every m, which again implies that (29) is satisfied for every $1 < g \le \sqrt{m}$. This case is not interesting.

If $\varepsilon = 1$, i.e. $m = n = g$, then (30) implies that for $g \ge 2.195 \times 10^{25}$ we have only the trivial solutions. This case is completely solved in [13], as we mentioned before.

Let us suppose $\varepsilon_0 \in (0.5, 1)$ and $M_0(\varepsilon_0) = m_0$. Then

$$m^{\varepsilon_0} < G \cdot m^{(1-\varepsilon_0)} \log^{10}\left(2m^{(1-\varepsilon_0)}\right)$$

implies $m < m_0$. So, for $g = m^{\varepsilon}$, $m \ge m_0$ and $g \ge m^{\varepsilon_0} \ge m_0^{\varepsilon_0}$ we have only the trivial solutions $(\pm 1, 0)$ and $(0, \pm 1)$ of equation (1). Let us give some special cases:

- if $\varepsilon_0 = 0.999$, which implies $m_0 < 10^{26}$, then for $m \ge 10^{26}$ and $\varepsilon \ge 0.999$, i.e.

$$g \ge m^{0.999} = \max\left\{m^{0.999},\ n^{0.999}\right\} \ge 10^{25.974} > (m_0)^{0.999},$$

  we have only the trivial solutions.

- Similarly we find that for: $g \ge m^{0.99} \ge 10^{28.71}$, $g \ge m^{0.9} \ge 10^{42.3}$, $g \ge m^{0.8} \ge 10^{56.8}$, $g \ge m^{0.7} \ge 10^{80.5}$, $g \ge m^{0.6} \ge 10^{152.4}$, $g \ge m^{0.51} \ge 10^{1591.2}$, $g \ge m^{0.501} \ge 10^{18453.834}$ we have only the trivial the solutions.

2) If we suppose that $1 \leq m < n$ and $g = \gcd(m, n) > 1$, then in the similar manner, we can find that for every $\varepsilon_0 \in (0.5, 1)$ if

$$g \geq n^{\varepsilon_0} = \max\{m^{\varepsilon_0},\ n^{\varepsilon_0}\},$$

and $n \geq n_0$, where $n_0 = N_0(\varepsilon_0)$ and $N_0(\varepsilon)$ is a strictly decreasing function implicitly given by

$$N_0^{\varepsilon} = 1.875 \times 10^{27} \cdot N_0^{(1-\varepsilon)} \cdot \log^{10}\left(2M_0^{(1-\varepsilon)}\right)$$

with $N_0(\varepsilon) \geq 2$, then equation (1) has only the trivial solutions. In particular, we find that for: $g \geq n^{0.999} \geq 10^{26.973}$, $g \geq n^{0.99} \geq 10^{29.7}$, $g \geq n^{0.9} \geq 10^{43.2}$, $g \geq n^{0.8} \geq 10^{56.8}$, $g \geq n^{0.7} \geq 10^{81.2}$, $g \geq n^{0.6} \geq 10^{153}$, $g \geq n^{0.51} \geq 10^{1600.38}$, $g \geq n^{0.501} \geq 10^{18454.836}$ we have only the trivial solutions. (For details see [13, Section 5.4.3])

Let us remark that $N_0(\varepsilon) > M_0(\varepsilon)$ for every $\varepsilon \in (0.5, 1)$, so the combination of **1)** and **2)** finishes the proof of Theorem 1.

8 Bound for the number of the solutions

In [4], Bennett proved that the system of simultaneous Pell equations

$$x^2 - az^2 = 1, \qquad y^2 - bz^2 = 1 \tag{32}$$

where a and b are distinct positive integers, possess at most three solutions in positive integers (x, y, z). Recently, Yuan [31] proved that if $\max(a, b) > 1.4 \cdot 10^{57}$, then the system (32) possess at most two solutions.

In this section we follow Bennett [4] in proving that system (13) and (14) has at most 7 solutions in positive integers (V, Z, U).

Let $m \geq 0$, $n \geq 2$ and $V = V_{k_i}$, $Z = Z_{l_i}$, $U = U_{k_i} = T_{l_i}$ be positive solutions of system (13) and (14) for $i = 1, 2, 3$, where $U_{k_1} < U_{k_2} < U_{k_3}$. We consider the determinant Δ defined as follows

$$\Delta = \begin{vmatrix} V_{k_1} & Z_{l_1} & U_{k_1} \\ V_{k_2} & Z_{l_2} & U_{k_2} \\ V_{k_3} & Z_{l_3} & U_{k_3} \end{vmatrix} = \begin{vmatrix} V_{k_1} - \sqrt{m^2+2}\,U_{k_1} & Z_{l_1} - \sqrt{n^2-2}\,T_{l_1} & U_{k_1} \\ V_{k_2} - \sqrt{m^2+2}\,U_{k_2} & Z_{l_2} - \sqrt{n^2-2}\,T_{l_2} & U_{k_2} \\ V_{k_3} - \sqrt{m^2+2}\,U_{k_2} & Z_{l_3} - \sqrt{n^2-2}\,T_{l_3} & U_{k_3} \end{vmatrix}.$$

If we expand Δ along the third column and define $\alpha_i = V_{k_i} + \sqrt{m^2+2}\,U_{k_i}$ and $\beta_i = Z_{l_i} + \sqrt{n^2-2}\,T_{l_i}$, as in [4, Lemmas 6.1 and 6.2], we find

$$\Delta = \frac{1}{\sqrt{(m^2+2)(n^2-2)}\,U_{k_1}U_{k_2}U_{k_3}} \sum_{1 \leq i \leq 3} U_{k_i}^2 \delta_i$$

where

$$\begin{aligned}
\delta_1 &= 2\left(\alpha_3^{-2}-\alpha_2^{-2}\right)-2\left(\beta_2^{-2}-\beta_3^{-2}\right)+4\left(\alpha_2^{-2}\beta_3^{-2}-\alpha_3^{-2}\beta_2^{-2}\right)\\
\delta_2 &= 2\left(\alpha_1^{-2}-\alpha_3^{-2}\right)-2\left(\beta_3^{-2}-\beta_1^{-2}\right)+4\left(\alpha_3^{-2}\beta_1^{-2}-\alpha_1^{-2}\beta_3^{-2}\right)\\
\delta_3 &= 2\left(\alpha_2^{-2}-\alpha_1^{-2}\right)-2\left(\beta_1^{-2}-\beta_2^{-2}\right)+4\left(\alpha_1^{-2}\beta_2^{-2}-\alpha_2^{-2}\beta_1^{-2}\right)
\end{aligned}$$

We can prove $|\delta_i| < \frac{1}{U_{k_1}^2} + \frac{1}{9U_{k_1}^4}$ for $i = 1, 2, 3$, which implies that

$$\begin{aligned}
|\Delta| &\leq \frac{1}{\sqrt{(m^2+2)(n^2-2)}\,U_{k_1}U_{k_2}U_{k_3}}\sum_{1\leq i\leq 3} U_{k_i}^2\,|\delta_i|\\
&< \frac{10}{3\sqrt{(m^2+2)(n^2-2)}\cdot U_{k_1}^3 U_{k_2}} U_{k_3}.
\end{aligned}$$

Since Δ is an integer and $\Delta \neq 0$, it follows that

$$U_{k_3} > \frac{3\sqrt{(m^2+2)(n^2-2)}}{10}\,U_{k_1}^3 U_{k_2}.$$

If $m \geq n$, except for $m = n = 2$, we have

$$U_{k_3} > 1.4\cdot U_{k_1}^3 U_{k_2}. \tag{33}$$

From recurrences (16) we obtain

$$U_5 < 8\left(m^2+1\right)^3 U_2 < 1.4\cdot\left(2m^2+1\right)^3 U_2 = 1.4\cdot U_1^3 U_2,$$

which implies $k_3 \geq 6$. If $m = n = 2$ we have only trivial positive solution $(V, Z, U) = (2, 2, 1)$. For details see [13, Proposition 8.8].
If $m < n$, except for $n = 2$, using $U_{k_i} = T_{l_i}$, we find $T_{l_3} > 1.9 \cdot T_{l_1}^3 T_{l_2}$. Similarly, from recurrences (18), we obtain

$$T_5 < 8\left(n^2-1\right)^3 T_2 < 1.9\cdot\left(2n^2-1\right)^3 T_2 = 1.9\cdot T_1^3 T_2,$$

which implies $l_3 \geq 6$. If $m = 0$ and $n = 2$, it's easy to show that we have only trivial non-negative solution $(V, Z, U) = (0, 2, 1)$. If $m = 1$ and $n = 2$ we have two positive solutions as we mentioned at the end of Section 5.

Let $M = \max\left\{m^2+2,\ n^2-2\right\}$ and $(V_{k_i}, Z_{l_i}, U_{k_i})$, $(V_{k_j}, Z_{l_j}, U_{k_j})$ be two positive solutions of (13) and (14) which satisfying $U_{k_i} < U_{k_j}$. We will use the following theorem of Bennett [4, Theorem 3.2]

Theorem 3 *If a_i, p_i, q and N are integers for $0 \leq i \leq 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \leq j \leq 2$, q nonzero and $N > M_1^9$, where*

$$M_1 = \max_{0 \leq i \leq 2} \{|a_i|\},$$

then we have

$$\max_{0 \leq i \leq 2} \left\{ \left| \sqrt{1 + \frac{a_i}{N}} - \frac{p_i}{q} \right| \right\} > (130 N \gamma)^{-1} q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33 N \gamma)}{\log\left(1.7 N^2 \prod_{0 \leq i < j \leq 2} (a_i - a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2 - a_0)^2 (a_2 - a_1)^2}{2a_2 - a_0 - a_1} & \text{if } a_2 - a_1 \geq a_1 - a_0, \\ \frac{(a_2 - a_0)^2 (a_1 - a_0)^2}{a_1 + a_2 - 2a_0} & \text{if } a_2 - a_1 < a_1 - a_0. \end{cases}$$

We will apply Theorem 3 with $a_0 = 0$, $a_1, a_2 \in \left\{2\left(m^2+2\right),\ 2\left(n^2-2\right)\right\}$, $M_1 = 2M$, $N = \left(m^2+2\right)\left(n^2-2\right)U_{k_i}^2$, $q = \left(m^2+2\right)\left(n^2-2\right)U_{k_i}U_{k_j}$, $p_1 = \left(m^2+2\right)V_{k_i}V_{k_j}$, $p_2 = \left(n^2-2\right)Z_{l_i}Z_{l_j}$. Since $N \geq M_1 U_{k_i}^2$, we may apply Theorem 3 if $U_{k_i} > 16M^4$. We have (see [4, Lemma 6.2])

$$\max\left\{\left\{\left|\sqrt{1+\frac{a_1}{N}} - \frac{p_1}{q}\right|,\ \left|\sqrt{1+\frac{a_2}{N}} - \frac{p_2}{q}\right|\right\}\right\} < \frac{2}{U_{k_j}^2}.$$

Also $N\gamma \leq 8 \cdot M^5 U_{k_i}^2$ and

$$N^2 \prod_{0 \leq i < j \leq 2} (a_i - a_j)^{-2} \geq \frac{U_{k_i}^4}{2^8 \cdot M^2}.$$

So, for $m \geq n$ and $m \geq 3$, $U_{k_i} > 33.88 \cdot M^6 > 6 \times 10^7$ implies $\lambda < 1.8635$. It follows from Theorem 3 that

$$(130 \cdot 8 M^5 U_{k_i}^2)^{-1} (M^2 U_{k_i} U_{k_j})^{-1.8635} < \frac{2}{U_{k_j}^2},$$

which implies

$$U_{k_j} < 2080^{7.327} M^{63.935} U_{k_i}^{28.305} < 1.0129 \times 10^8 \cdot U_{k_i}^{38.97} < U_{k_i}^{40}. \tag{34}$$

For $m < n$, except for $n = 2$, we can prove, using $U_{k_i} = T_{l_i}$ and $U_{k_j} = T_{l_j}$, that $T_{l_i} > 148.6 \cdot M^6 > 1.7 \times 10^7$ implies $\lambda < 1.842$. So, it follows from Theorem 3 that $T_{l_j} < T_{l_i}^{34}$.

Let $(V_{k_i}, Z_{l_i}, U_{k_i})$, where $U_{k_i} = T_{l_i}$, be sequence of all positive solutions of (13) and (14) with $U_{k_i} < U_{k_j}$ for $i < j$. Let $m \geq n$. Since $U_{k_3} \geq U_6 > 33.88 \cdot M^6$, except for $m = n = 2$, we have $U_{k_i} < U_{k_3}^{40}$ for $i \geq 4$. Using (33) successively, we obtain: $U_{k_5} > U_{k_4} U_{k_3}^3 > U_{k_3}^4$, $U_{k_6} > U_{k_5} U_{k_4}^3 > U_{k_3}^7$, $U_{k_7} > U_{k_6} U_{k_5}^3 > U_{k_3}^{19}$, $U_{k_8} > U_{k_7} U_{k_6}^3 > U_{k_3}^{40}$. So we have, by (34), a contradiction, hence k_8 doesn't exist.

For $m < n$, except for $n = 2$, since $T_{l_3} \geq T_6 > 148.6 \cdot M^6$, we have $T_{l_i} < T_{l_3}^{34}$ for $i \geq 4$. Now, we obtain, as before, that l_8 doesn't exist. Therefore we proved

Theorem 4 *System (13) and (14) for all* $m \geq 0$ *and* $n \geq 2$ *possess at most* 7 *solutions in positive integers* (V, Z, U) .

Remark 3 The result from Theorem 4 on the system of Pellian equations (13) and (14) is weaker than the corresponding Bennett's result on systems of Pell equations. This is mainly because, at present, there is no analogon of Bennett's doubly exponential gap principle [4, Lemma 2.2] for systems of Pellian equations. The difference between Pell and Pellian equations is also the reason why we had to use linear forms in three logarithms in Section 5, while in [4, Section 4] linear forms in two logarithms were used.

Acknowledgements: The author would like to thank Professor Andrej Dujella for helpful suggestions.

References

[1] A. Baker, Contributions to the theory of Diophantine equations I. On the representation of integers by binary forms, *Philos. Trans. Roy. Soc. London Ser. A* **263** (1968), 173–191.

[2] A. Baker and H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser. (2)* **20** (1969), 129–137.

[3] A. Baker and G. Wüstholz, Logarithmic forms and group varieties, *J. Reine Angew. Math.* **442** (1993), 19–62.

[4] M. A. Bennett, On the number of solutions of simultaneous Pell equations, *J. Reine Angew. Math.* **498** (1998), 173–199.

[5] Yu. Bilu, G. Hanrot, Solving Thue equations of high degree, *J. Number Theory*, **60** (1996), 373–392.

[6] J. H. Chen, P. M. Voutier, Complete solution of the Diophantine equation $X^2 + 1 = dY^4$ and a related family of Thue equations, *J. Number Theory* **62** (1996), 273–292.

[7] T.W. Cusick, The diophantine equation $X^4 - kX^2Y^2 + Y^4 = 1$, *Arch Math.* **59** (1992), 345–347.

[8] A. Dujella, Complete solution of a family of simultaneous Pellian equations, *Acta Math. Inform. Univ. Ostraviensis* **6** (1998), 59–67.

[9] A. Dujella, An absolute bound for the size of Diophantine m-tuples, *J. Number Theory* **89** (2001), 126-150.

[10] A. Dujella and B. Jadrijević, A parametric family of quartic Thue equations, *Acta Arith.* , **101** (2002), 159–170.

[11] A. Dujella and A. Pethő, A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.* (2) **49** (1998), 291–306.

[12] C. Heuberger, A. Pethő, R. F. Tichy, Complete solution of parametrized Thue equations, *Acta Math. Inform. Univ. Ostraviensis* **6** (1998), 93–113.

[13] B. Jadrijević, *A two-parametric family of quartic Thue equations* (in Croatian), PhD thesis, University of Zagreb, 2001.

[14] G. Lettl and A. Pethő, Complete solution of a family of quartic Thue equations, *Abh. Math. Sem. Univ. Hamburg*, **65** (1995), 365–383.

[15] W. Ljunggren, Über die Gleichung $x^4 - Dy^2 = 1$, *Arch. Math. Naturvid.*, **45** (1942), 1–12.

[16] M. Mignotte, A. Pethő, R. Roth, Complete solutions of quartic Thue and index form equations, *Math. Comp.* **65** (1996), 341–354.

[17] L. J. Mordell, "Diophantine Equations", Academic Press, London, 1969.

[18] T. Nagell, Sur quelques questions dans la théorie des corps biquadratiques, *Ark. Mat.* **4** (1961), 347–376.

[19] T. Nagell, "Introduction to the Number Theory", Almqvist, Stockholm, Wiley, New York, 1951.

[20] A. Pethő, Complete solutions to families of quartic Thue equations, *Math. Comp.* **57** (1991), 777–798.

[21] A. Pethő, R. Schulenberg, Effectives Lösen von Thue Gleichungen, *Publ. Math. Debrecen* **34** (1987), 189–196.

[22] A. Pethő, R. T. Tichy, On two-parametric quartic families of Diophantine problems, *J. Symbolic Comput.* **26** (1998), 151–171.

[23] E. Thomas, Complete solutions to a family of cubic Diophantine equations, *J. Number Theory* **34** (1990), 235–250.

[24] A. Thue, Über Annäherungswerte algebraischer Zahlen, *J. Reine Angew. Math.* **135** (1909), 284–305.

[25] A. Togbé, On the solutions of a family of quartic Thue equations, *Math. Comp.* **69** (2000), 839–849.

[26] N. Tzanakis, Explicit solution of a class of quartic Thue equations, *Acta Arith.* **64** (1993), 271–283.

[27] N. Tzanakis, B. M. M. de Weger, On the practical solution of the Thue equation, *J. Number Theory* **31** (1989), 99–132.

[28] I. Wakabayashi, On a family of quartic Thue inequalities, *J. Number Theory* **66** (1997), 70–84.

[29] I. Wakabayashi, On a family of quartic Thue inequalities, II, *J. Number Theory* **80** (2000), 60–88.

[30] G. Walsh, A note a theorem of Ljunggren and the Diophantine equations $x^2 - kxy^2 + y^4 = 1, 4$, *Arch Math.* **73,** No.2**,** (1999), 119–125.

[31] P. Yuan, On the number of solutions of simultaneous Pell equations, *Acta Arith.* **101** (2002), 215-221.

FESB, University of Split, R. Boškovića bb, 21000 Split, Croatia
E-mail address: borka@fesb.hr