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Harmonijske oscilacije

~ nastupno predavanje ~

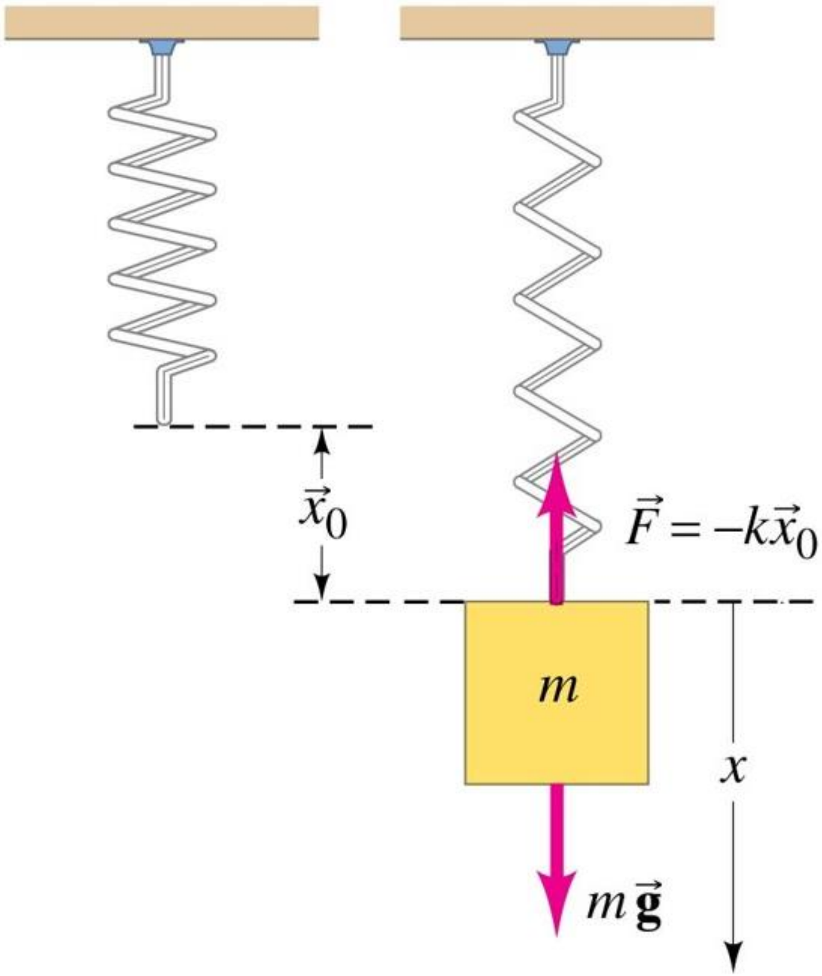
PMFST, 6. 10. 2017.

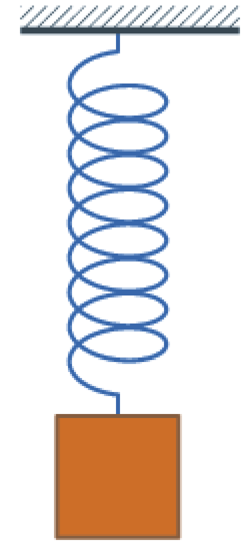
» Oscilacije

- **Titranje** (osciliranje) jest proces (gibanja ili promjene stanja) koji se u manjoj ili većoj mjeri ponavlja u vremenu.
  - mehanička
  - elektromagnetska
  - elektromehanička
- Titranje se naziva **periodičnim** ako se vrijednosti promjenjivih fizikalnih veličina ponavljaju u jednakim vremenskim intervalima.
- Periodično titranje naziva se **harmonijskim titranjem** ako se može opisati funkcijama sinus i kosinus.
- Koliko su titranja prisutna oko nas?

≫ Tijelo obješeno o oprugu

- PP: opruga elastična +nema sila otpora
- ravnotežni položaj, oscilacije





$$\frac{d^2x(t)}{dt^2} = -\omega_0^2 \cdot x(t)$$

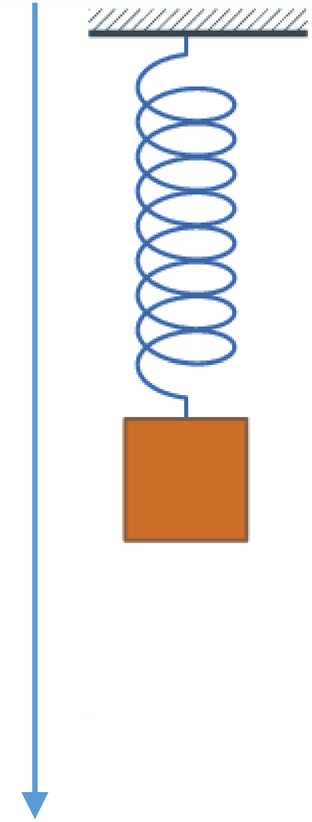
$$\omega_0 = \sqrt{k/m}$$

Tijelo obješeno o oprugu

- PP: opruga elastična +nema sila otpora

$$\frac{d^2x(t)}{dt^2} = -\omega_0^2 \cdot x(t) \qquad \omega_0 = \sqrt{k/m}$$

- obična homogena linearna diferencijalna jednadžba 2. reda s konstantnim koeficijentima



» Matematička podloga

- Uvod u homogene linearne diferencijalne jednadžbe

$$a_0 x^{(n)} + a_1 x^{(n-1)} + \cdots + a_{n-1}\dot{x} + a_n x = 0$$

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$

$$x = \sum_{i=1}^{n} C_i x^i$$

$(\lambda_i \in \mathbb{R})$ kratnosti p

$$x_1^i = \exp(\lambda_i t)$$
$$x_2^i = t\exp(\lambda_i t)$$
$$\cdots\cdots$$
$$x_p^i = t^{p-1}\exp(\lambda_i t)$$

$(\lambda_i \in \mathbb{R})$ kratnosti 1

$$x^i = \exp(\lambda_i t)$$

$(\lambda_i = a \pm \beta\sqrt{-1})$ kratnosti s

$$x_1^i = \exp(\alpha t)\cos(\beta t)$$
$$x_2^i = t\exp(\alpha t)\cos(\beta t)$$
$$\cdots\cdots$$
$$x_s^i = t^{s-1}\exp(\alpha t)\cos(\beta t)$$
$$x_{s+1}^i = \exp(\alpha t)\sin(\beta t)$$
$$x_{s+2}^i = t\exp(\alpha t)\sin(\beta t)$$
$$\cdots\cdots$$
$$x_{2s}^i = t^{s-1}\exp(\alpha t)\sin(\beta t)$$

$(\lambda_i = a \pm \beta\sqrt{-1})$ kratnosti 1

$$x_1^i = \exp(\alpha t)\cos(\beta t)$$
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≫ Tijelo obješeno o oprugu

- PP: opruga elastična +nema sila otpora

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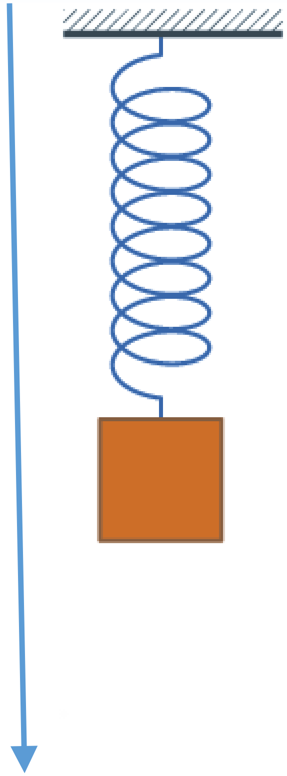


- obična homogena linearna diferencijalna jednadžba 2. reda s konstantnim koeficijentima čije je opće rješenje

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

$$C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \cos(\omega_0 t + \varphi)$$

$$C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A \cos(\omega_0 t) \cos(\varphi) - A \sin(\omega_0 t) \sin(\varphi)$$

$$C_1 = A \cos(\varphi)$$

$$C_2 = -A \sin(\varphi)$$

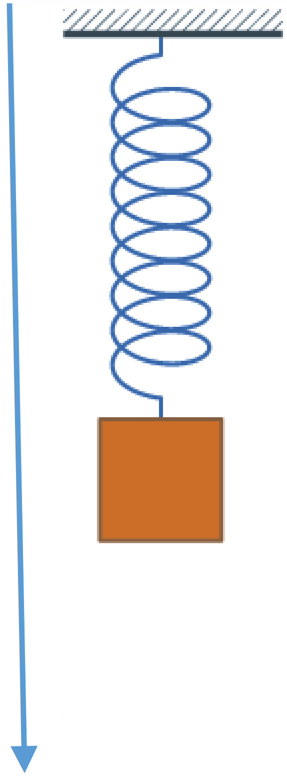
$$\boxed{x(t) = A \cos(\omega_0 t + \varphi)}$$

≫ Tijelo obješeno o oprugu

- PP: opruga elastična +nema sila otpora

$$\frac{d^2x(t)}{dt^2} = -\omega_0^2 \cdot x(t) \qquad \omega_0 = \sqrt{k/m}$$

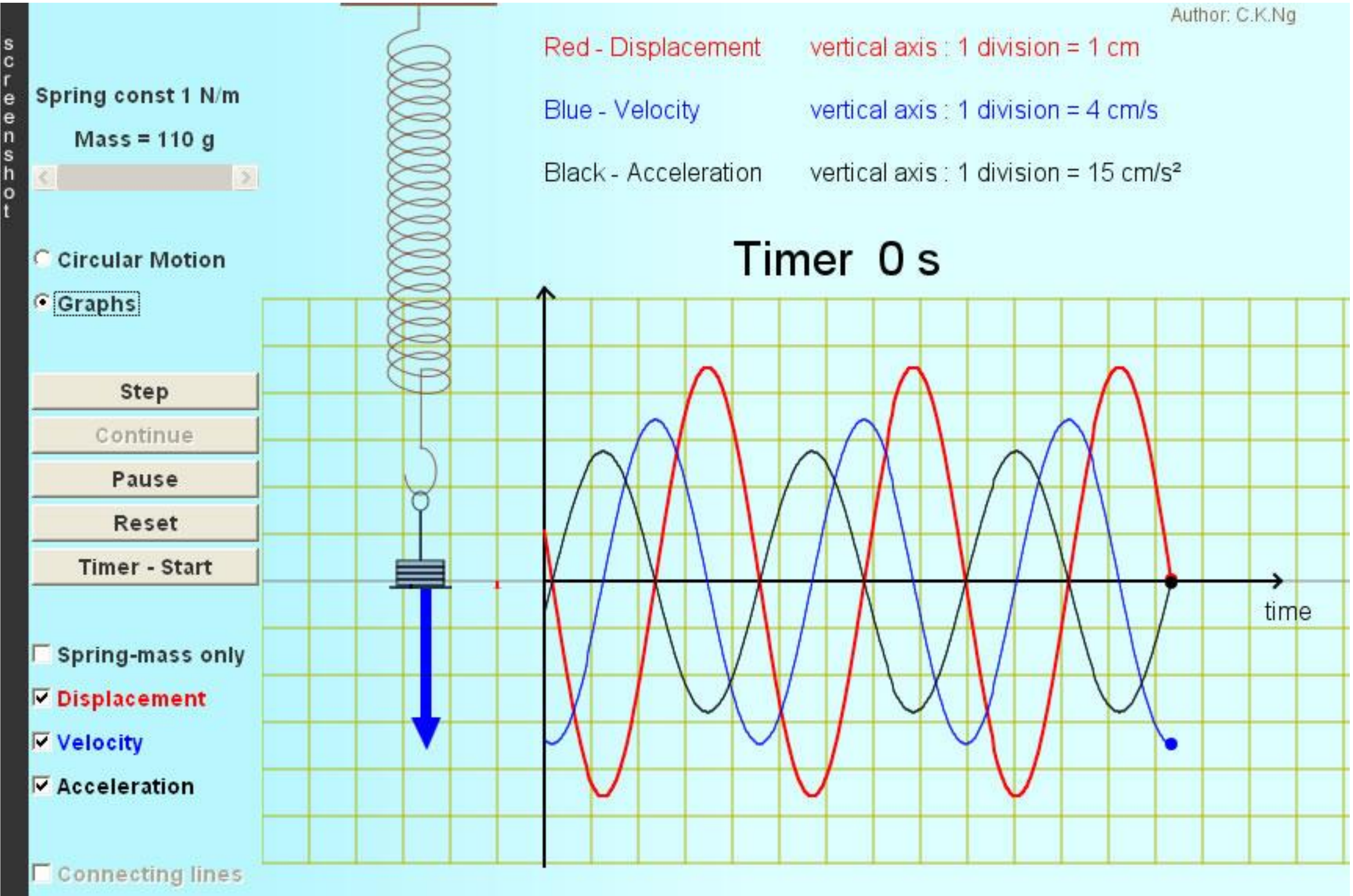
$$x(t) = A\cos(\omega_0 t + \varphi)$$



- A = amplituda
- x = pomak iz ravnotežnog položaja (elongacija)
- $\omega_0 = 2\pi f$ = kružna (kutna) frekvencija
- $\Phi = \omega_0 t + \varphi$ = faza
- φ = početna faza
- period $T = \dfrac{2\pi}{\omega_0}$
- frekvencija $f = T^{-1}$

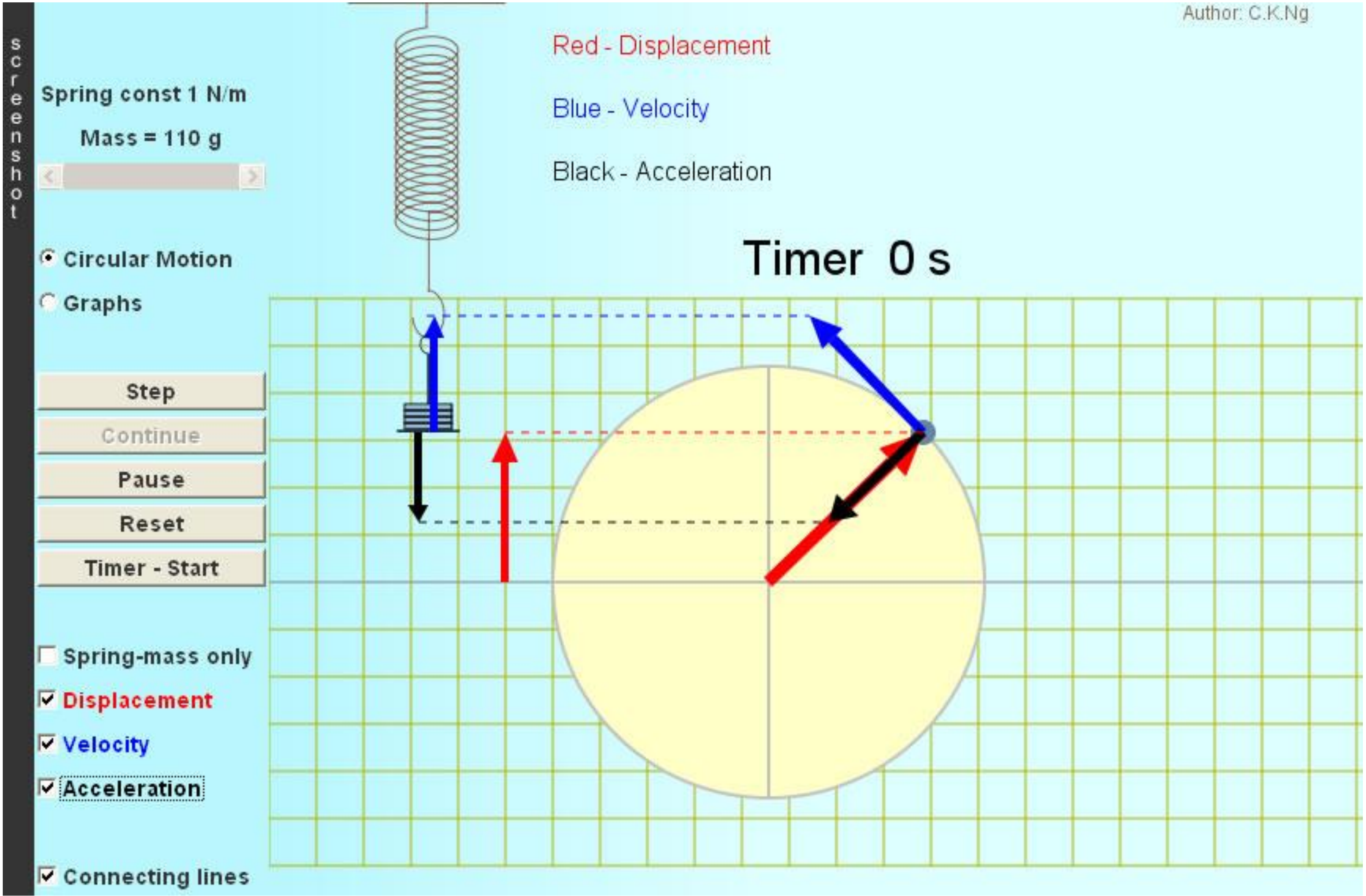
Simulacija

- URL: http://ngsir.netfirms.com/englishhtm/SpringSHM.htm



≫ Analogija s kružnim gibanjem

- URL: http://ngsir.netfirms.com/englishhtm/SpringSHM.htm



≫ Harmonijsko gibanje (prikaz rješenja)

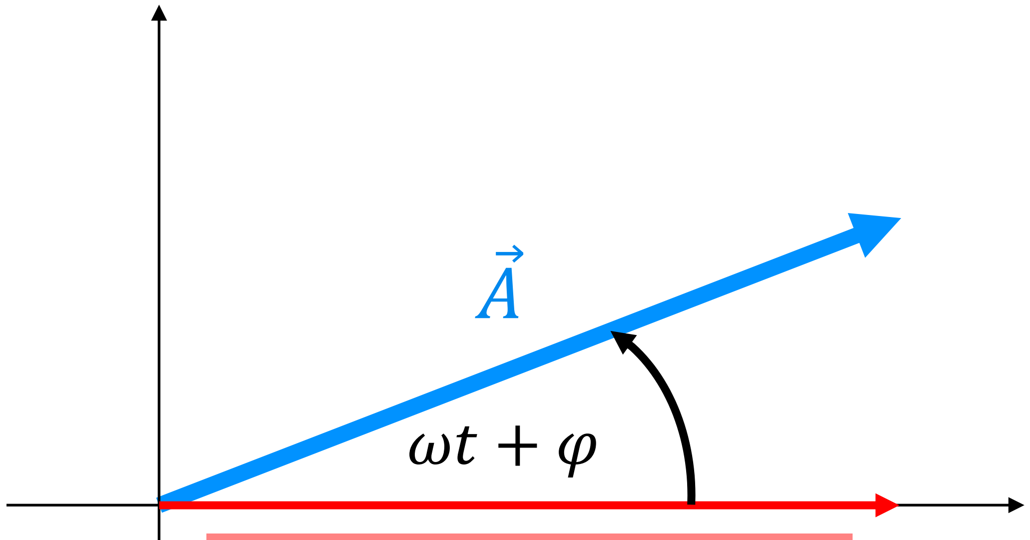
- trigonometrijskim funkcijama

$$x(t) = A\cos(\omega t + \varphi)$$

- u eksponencijalnom obliku

$$x(t) = \mathrm{Re}\,\tilde{x}(t), \qquad \tilde{x}(t) = Ae^{i(\omega t+\varphi)}$$

- fazorima (rotirajućim vektorima)



$$x(t) = A\cos(\omega_0 t + \varphi)$$

≫ Matematičko njihalo

- PP: masa niti zanemariva, a obješeno tijelo = točkasta masa

$$\vec{F} = mg\sin\theta\ (-\hat{s})$$

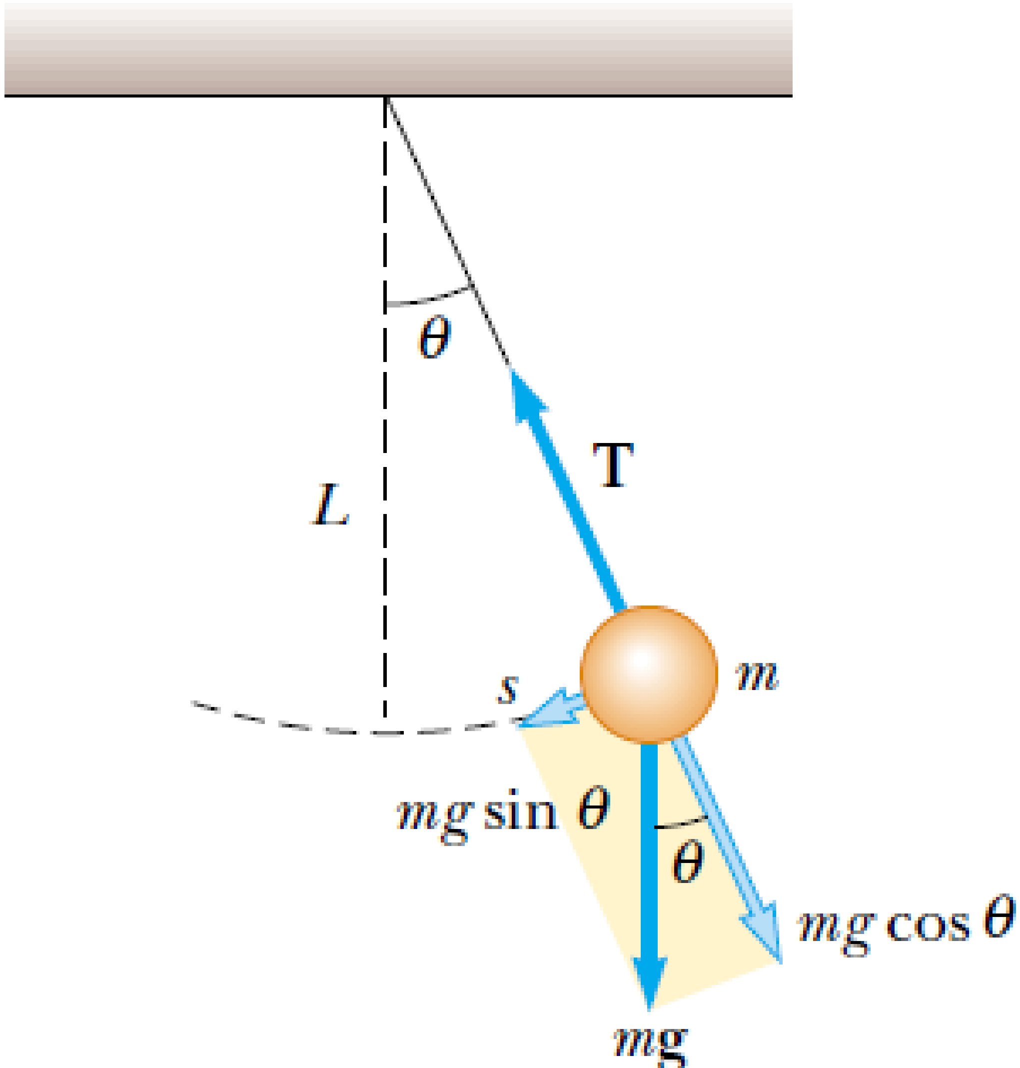
$$m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2}\hat{s} = mg\sin\theta\ (-\hat{s})$$

$$s = L\theta \qquad \sin\theta \approx \theta$$

$$L\ddot{\theta} = -g\theta$$

$$\ddot{\theta} = -\frac{g}{L}\theta$$

$$\omega_0 = \sqrt{g/L}$$



≫ Fizikalno njihalo

PP: kruto tijelo koje može rotirati oko O (z osi)

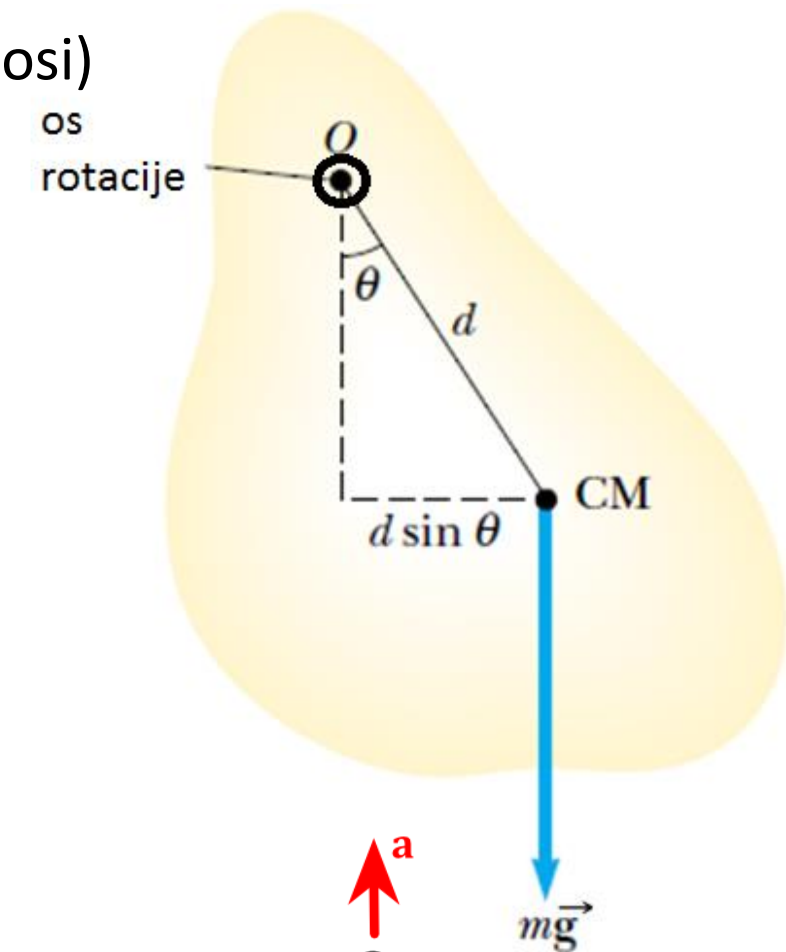
$$\vec{M} = \vec{r} \times \vec{F}$$

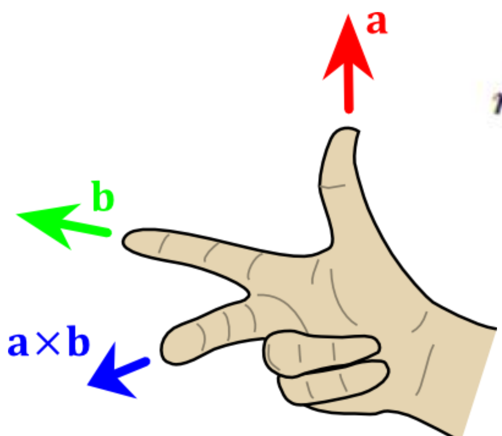
$$I \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \hat{z} = mgd \sin\theta \;\; (-\hat{z})$$

$$\sin\theta \approx \theta$$

$$\ddot{\theta} = -\frac{mgd}{I}\theta$$

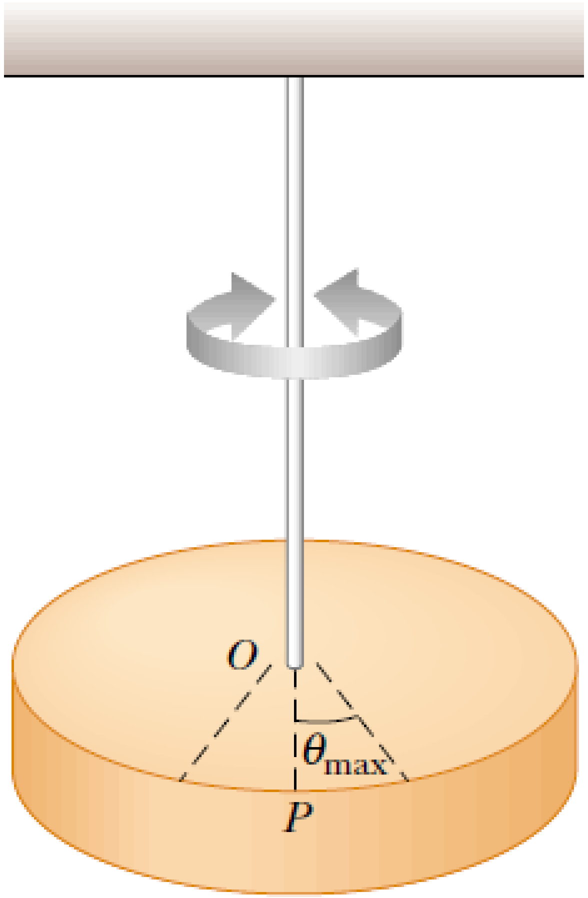
$$\omega_0 = \sqrt{\frac{mgd}{I}}$$





≫ Torzijski oscilator

PP: rotiranje diska proizvodi tangencijalnu silu na valjak što dovodi do torzije valjka o koji je obješen disk



$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -k\theta$$

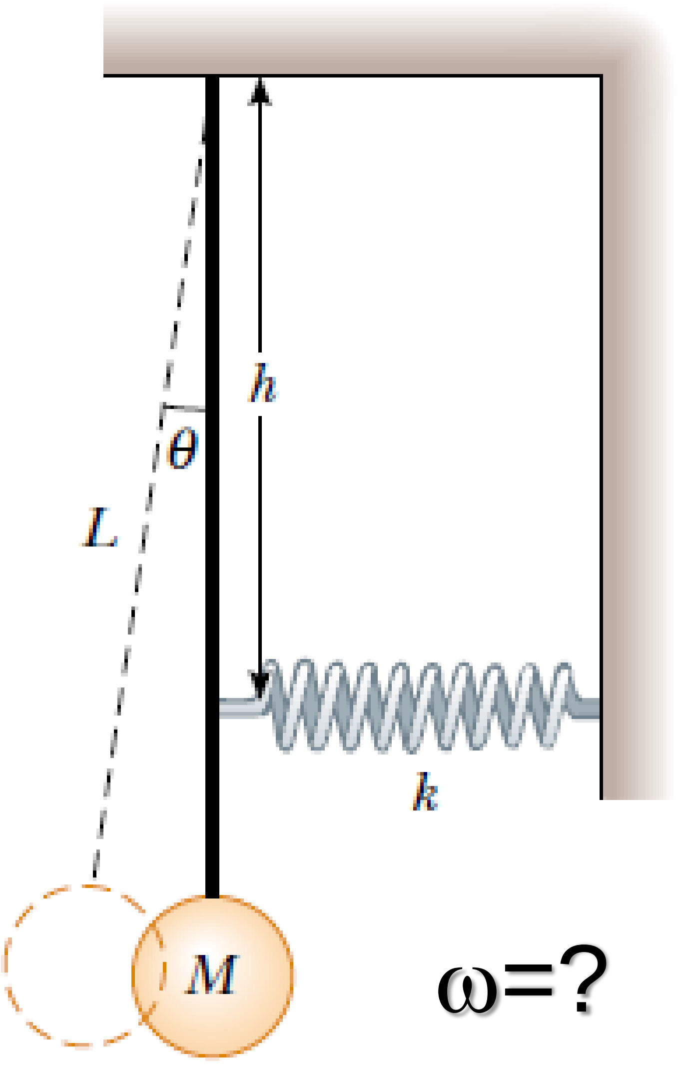
$$\ddot{\theta} = -\frac{k}{I}\theta$$

$$\omega_0 = \sqrt{\frac{k}{I}}$$

≫ Kombinirani oscilator

PP: sile otpora zanemarive

$$\omega = \sqrt{\frac{MgL + kh^2}{ML^2}}$$



» Energija jednostavnog harmonijskog oscilatora

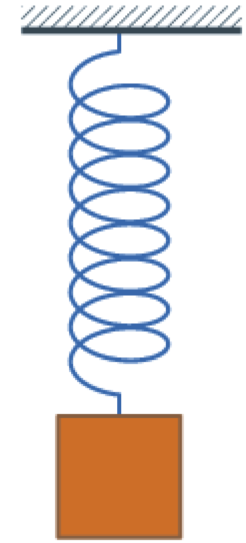
$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

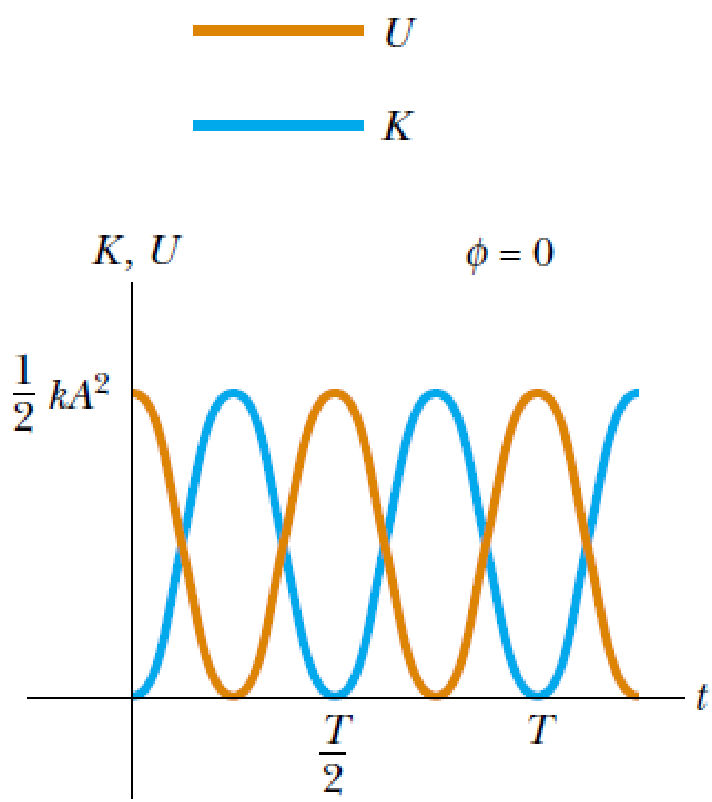
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$

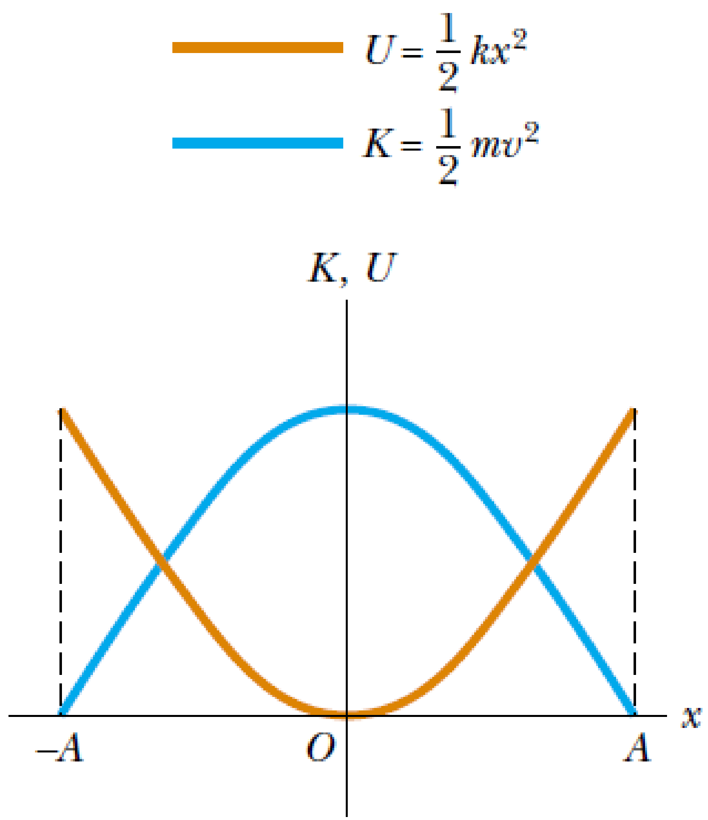
$$E = \frac{1}{2}kA^2$$

$$\frac{d^2x}{dt^2} = -\omega^2 x \qquad \omega^2 = \frac{k}{m}$$

$$x(t) = A\cos(\omega t + \phi)$$







Analogija s LC krugom

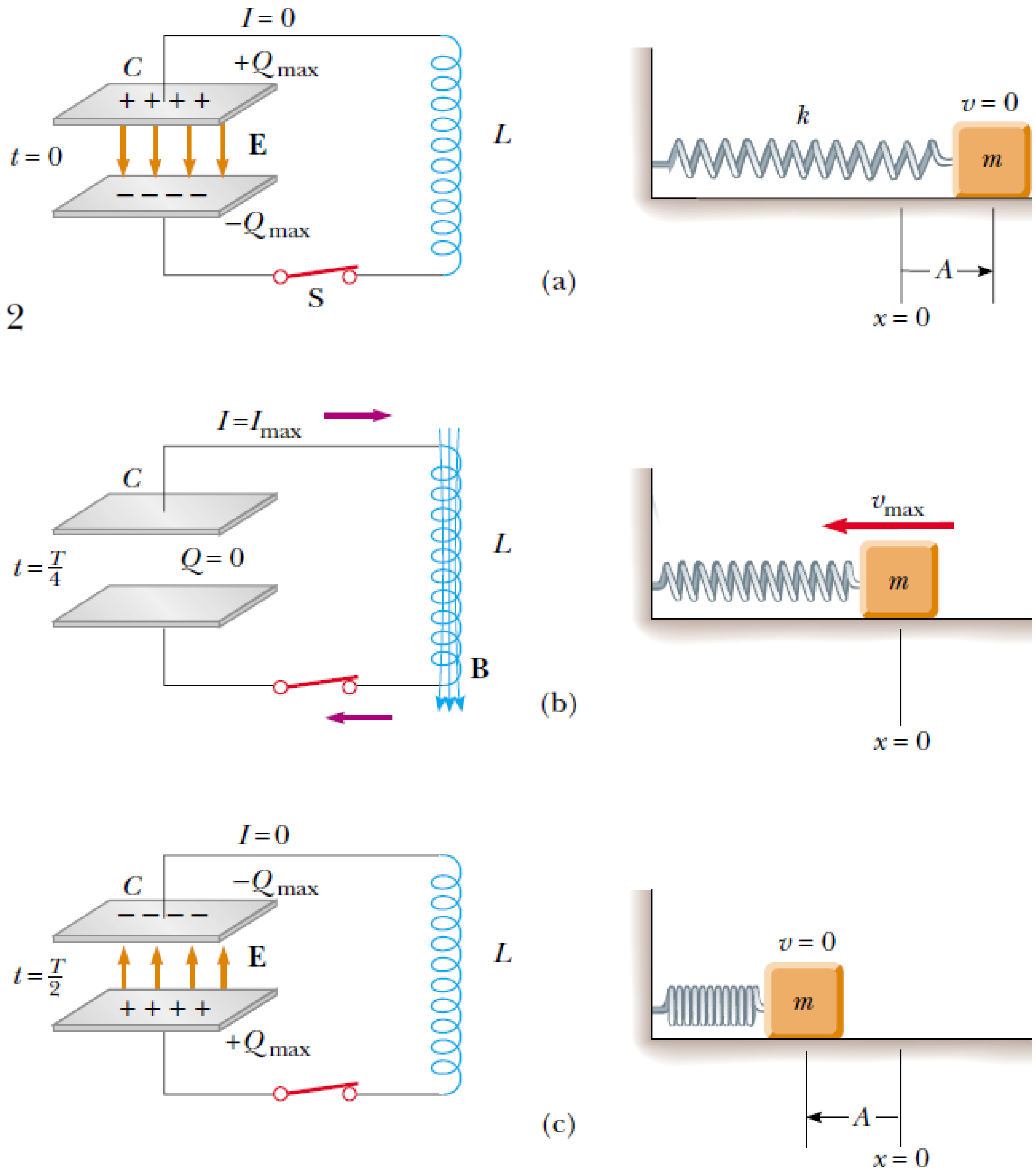
$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2$$

$$\frac{dU}{dt} = 0$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$$

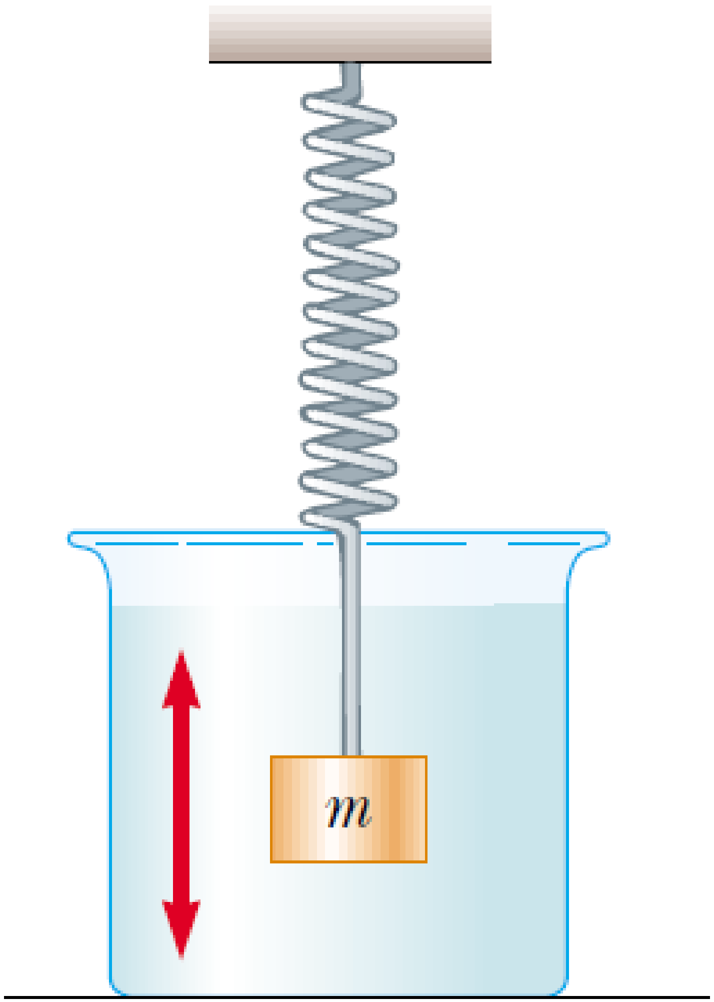
$$\omega = \frac{1}{\sqrt{LC}}$$

$$Q = Q_{\max}\cos(\omega t + \phi)$$



≫ Prigušeni harmonijski oscilator

- PP: sila gušenja proporcionalna brzini



$$\vec{F} = -k\vec{x} \qquad \overrightarrow{F_{TR}} = -b\vec{v}$$

$$\frac{d^2x}{dt^2} + 2\delta\frac{dx}{dt} + \omega_0^2 x = 0$$

$$x(t) = A(t)\cos(\omega t + \varphi)$$

$$A(t) = Ae^{-\delta t}$$

$$\omega_0 = \sqrt{k/m}$$

$$\omega = \sqrt{\omega_0^2 - \delta^2}$$

$$\delta = \frac{b}{2m} = \frac{\Gamma}{2}$$

$$\varphi = \varphi_{POČETNI}$$

≫ Prisilne harmonijske oscilacije

$\vec{F} = -k\vec{x}$ | $\overrightarrow{F_{TR}} = -b\vec{v}$ | $\vec{F} = \vec{F}_0 \cos(\omega t)$

$$\frac{d^2x}{dt^2} + 2\delta\frac{dx}{dt} + \omega_0^2 x = A_0\cos(\omega t)$$

$$x(t) = A(\omega)\cos(\omega t + \varphi)$$

$$A(\omega) = \frac{A_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (2\delta\omega)^2}} \; ; \; A_0 = \frac{F_0}{m}$$

$$\omega_0 = \sqrt{k/m}$$

$$\omega = \omega_{VANJSKE\ SILE}$$

$$\delta = \frac{b}{2m} = \frac{\Gamma}{2}$$

$$\varphi = \mathrm{arctg}\frac{-2\delta\omega}{\omega_0^2 - \omega^2}$$