Partial differential equations

prof.dr.sc. Saša Krešić-Jurić

PMF–Split

Syllabus

1. Introduction
2. Fourier series
3. Classification of second order linear equations
4. Method of separation of variables
5. Wave equation
6. Heat equation
7. Laplace equation

Literature

1. Pinchover Y., Rubenstein J. *An Introduction to Partial Differential Equations*, Cambridge University Press, 2007.
2. Borthwick D. *Introduction to Partial Differential Equations*, Springer, 2018.
3. Guenther R.B., Lee J.W. *Partial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publications, 1996.

Syllabus

1. Introduction
2. Fourier series
3. Classification of second order linear equations
4. Method of separation of variables
5. Wave equation
6. Heat equation
7. Laplace equation

Literature

1. Pinchover Y., Rubenstein J. *An Introduction to Partial Differential Equations*, Cambridge University Press, 2007.
2. Borthwick D. *Introduction to Partial Differential Equations*, Springer, 2018.
3. Guenther R.B., Lee J.W. *Partial Differential Equations of Mathematical Physics and Integral Equations*, Dover Publications, 1996.

Introduction and basic definitions

- Physical phenomena which continously depend on spatial and time variables, such as wave motion and heat dissipation, are modelled by partial differential equations (PDE's)
- PDE's play an important role in physics, technology, biology, finance, etc.
- A partial differential equation describes a relation between an unkonwn function $u = u(x_1, x_2, \dots, x_n)$ and its partial derivatives. A general form for a PDE is can be written as

$$F\left(x, u, \frac{\partial u}{\partial x_i}, \dots, \frac{\partial^k u}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{j_k}}\right) = 0. \quad (1)$$

We introduce the notation

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad \text{etc.} \quad (2)$$

- The order of a partial differential equation is the order of the highest derivative appearing in the equation.
- PDE's are usually defined on an open connected set $\Omega \subseteq \mathbb{R}^n$.

- Physical phenomena which continously depend on spatial and time variables, such as wave motion and heat dissipation, are modelled by partial differential equations (PDE's)
- PDE's play an important role in physics, technology, biology, finance, etc.
- A partial differential equation describes a relation between an unkonwn function $u = u(x_1, x_2, \ldots, x_n)$ and its partial derivatives. A general form for a PDE is can be written as

$$F\left(x, u, \frac{\partial u}{\partial x_i}, \ldots, \frac{\partial^k u}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{j_k}}\right) = 0. \tag{1}$$

We introduce the notation

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad \text{etc.} \tag{2}$$

- The order of a partial differential equation is the order of the highest derivative appearing in the equation.
- PDE's are usually defined on an open connected set $\Omega \subseteq \mathbb{R}^n$.

- Physical phenomena which continously depend on spatial and time variables, such as wave motion and heat dissipation, are modelled by partial differential equations (PDE's)
- PDE's play an important role in physics, technology, biology, finance, etc.
- A partial differential equation describes a relation between an unkonwn function $u = u(x_1, x_2, \ldots, x_n)$ and its partial derivatives. A general form for a PDE is can be written as

$$F\left(x, u, \frac{\partial u}{\partial x_i}, \ldots, \frac{\partial^k u}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{j_k}}\right) = 0. \tag{1}$$

We introduce the notation

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad \text{etc.} \tag{2}$$

- The order of a partial differential equation is the order of the highest derivative appearing in the equation.
- PDE's are usually defined on an open connected set $\Omega \subseteq \mathbb{R}^n$.

Solutions of partial differential equations can be classfied as

1. classical,
2. weak and
3. distributional solutions.

Definition

A classical solution of a partial differential equation of order $k > 0$ on the set $\Omega \subseteq \mathbb{R}^n$ is a function $u \in C^k(\Omega)$ which satisfies the equation at every pont of the Ω.

A classical solution is also called a *strong* solution of the PDE.

Examples

1. Verify that $u = (x+y)^3$ i $u = \sin(x-y)$ are classical solutions of the equation

$$u_{xx} - u_{yy} = 0 \quad \text{na skupu} \quad \Omega = \mathbb{R}^2. \tag{3}$$

2. Verify that $u = \ln(x^2 + y^2)$ is a classical solution of the equation

$$u_{xx} + u_{yy} = 0 \quad \text{na skupu} \quad \Omega = \mathbb{R}^2 \setminus \{(0,0)\}. \tag{4}$$

Solutions of partial differential equations can be classfied as

1. classical,
2. weak and
3. distributional solutions.

Definition

A classical solution of a partial differential equation of order $k > 0$ on the set $\Omega \subseteq \mathbb{R}^n$ is a function $u \in C^k(\Omega)$ which satisfies the equation at every pont of the Ω.

A classical solution is also called a *strong* solution of the PDE.

Examples

1. Verify that $u = (x+y)^3$ i $u = \sin(x-y)$ are classical solutions of the equation

$$u_{xx} - u_{yy} = 0 \quad \text{na skupu} \quad \Omega = \mathbb{R}^2. \tag{3}$$

2. Verify that $u = \ln(x^2+y^2)$ is a classical solution of the equation

$$u_{xx} + u_{yy} = 0 \quad \text{na skupu} \quad \Omega = \mathbb{R}^2 \setminus \{(0,0)\}. \tag{4}$$

Solutions of partial differential equations can be classfied as

1. classical,
2. weak and
3. distributional solutions.

Definition

A classical solution of a partial differential equation of order $k > 0$ on the set $\Omega \subseteq \mathbb{R}^n$ is a function $u \in C^k(\Omega)$ which satisfies the equation at every pont of the Ω.

A classical solution is also called a *strong* solution of the PDE.

Examples

1. Verify that $u = (x+y)^3$ i $u = \sin(x-y)$ are classical solutions of the equation

$$u_{xx} - u_{yy} = 0 \quad \text{na skupu} \quad \Omega = \mathbb{R}^2. \tag{3}$$

2. Verify that $u = \ln(x^2+y^2)$ is a classical solution of the equation

$$u_{xx} + u_{yy} = 0 \quad \text{na skupu} \quad \Omega = \mathbb{R}^2 \setminus \{(0,0)\}. \tag{4}$$

Basic classification of partial differential equations

We can classify PDE's according to

1. the order of the equation,
2. linear vs. nonlinear equations.

- We say that the equation

$$F\left(x, u, \frac{\partial u}{\partial x_i}, \ldots, \frac{\partial^k u}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{j_k}}\right) = 0. \tag{5}$$

  is linear *linearna* if F if a linear function in the variables u and all its partial derivatives.

  In this case the coffeficients multplying the function u and its derivatives depend only on the independent variables $x_1, \ldots, x_n$.

- A PDE is *nonlinear* if it is not linear.

Basic classification of partial differential equations

We can classify PDE's according to

1. the order of the equation,
2. linear vs. nonlinear equations.

- We say that the equation

$$F\left(x, u, \frac{\partial u}{\partial x_i}, \dots, \frac{\partial^k u}{\partial x_{i_1}\partial x_{i_2}\dots\partial x_{j_k}}\right) = 0. \tag{5}$$

  is linear *linearna* if F if a linear function in the variables u and all its partial derivatives.

  In this case the coffeficients multplying the function u and its derivatives depend only on the independent variables $x_1, \dots, x_n$.

- A PDE is *nonlinear* if it is not linear.

Examples

1.
$$xu_x + yu_y = u, \quad \text{first order nonlinear PDE} \tag{6}$$

2.
$$u_t + u_{xxx} - 6uu_x = 0, \quad \text{nonlinear third order PDE} \tag{7}$$

3.
$$u_x^2 + u_y^2 = u, \quad \text{nonlinear first order PDE} \tag{8}$$

In mathematical physics and important role is played by partial differential equations of second order.

Examples

1.
$$xu_x + yu_y = u, \quad \text{first order nonlinear PDE} \tag{6}$$

2.
$$u_t + u_{xxx} - 6uu_x = 0, \quad \text{nonlinear third order PDE} \tag{7}$$

3.
$$u_x^2 + u_y^2 = u, \quad \text{nonlinear first order PDE} \tag{8}$$

In mathematical physics and important role is played by partial differential equations of second order.

Most physical phenomena are modelled by second order partial differential equations.

1. **The wave equation**

$$u_{tt} - c^2\nabla^2 u = 0, \quad u = u(x, y, z, t) \tag{9}$$

describes propagation of acoustic and electromagnetic waves in space.

2. **The heat equation**

$$u_t - k\nabla^2 u = 0, \quad u = u(x, y, z, t) \tag{10}$$

describes time evolution of temperature in heat conducting materials.

3. **The Laplace equation**

$$\nabla^2 u = 0, \quad u = u(x, y, z) \tag{11}$$

describes electric potential and also stationary distribution of temperature in a heat conducting material.

4. **The Schrödiger equation**

$$-\frac{\hbar}{2m}\nabla^2\psi + V(x, y, z)\psi = i\hbar\psi_t, \quad \psi = \psi(x, y, z, t) \tag{12}$$

describes evolution of the wave function ψ in quantum mechanics.

Classical equations of mathematical physics

Most physical phenomena are modelled by second order partial differential equations.

1. **The wave equation**

$$u_{tt} - c^2\nabla^2 u = 0, \quad u = u(x,y,z,t) \tag{9}$$

describes propagation of acoustic and electromagnetic waves in space.

2. **The heat equation**

$$u_t - k\nabla^2 u = 0, \quad u = u(x,y,z,t) \tag{10}$$

describes time evolution of temperature in heat conducting materials.

3. **The Laplace equation**

$$\nabla^2 u = 0, \quad u = u(x,y,z) \tag{11}$$

describes electric potential and also stationary distribution of temperature in a heat conducting material.

4. **The Schrödinger equation**

$$-\frac{\hbar}{2m}\nabla^2\psi + V(x,y,z)\psi = i\hbar\psi_t, \quad \psi = \psi(x,y,z,t) \tag{12}$$

describes evolution of the wave function ψ in quantum mechanics.

Most physical phenomena are modelled by second order partial differential equations.

1. **The wave equation**

$$u_{tt} - c^2\nabla^2 u = 0, \quad u = u(x,y,z,t) \tag{9}$$

describes propagation of acoustic and electromagnetic waves in space.

2. **The heat equation**

$$u_t - k\nabla^2 u = 0, \quad u = u(x,y,z,t) \tag{10}$$

describes time evolution of temperature in heat conducting materials.

3. **The Laplace equation**

$$\nabla^2 u = 0, \quad u = u(x,y,z) \tag{11}$$

describes electric potential and also stationary distribution of temperature in a heat conducting material.

4. **The Schrödinger equation**

$$-\frac{\hbar}{2m}\nabla^2\psi + V(x,y,z)\psi = i\hbar\psi_t, \quad \psi = \psi(x,y,z,t) \tag{12}$$

describes evolution of the wave function ψ in quantum mechanics.

Classical equations of mathematical physics

Most physical phenomena are modelled by second order partial differential equations.

1. **The wave equation**

$$u_{tt} - c^2 \nabla^2 u = 0, \quad u = u(x, y, z, t) \tag{9}$$

describes propagation of acoustic and electromagnetic waves in space.

2. **The heat equation**

$$u_t - k \nabla^2 u = 0, \quad u = u(x, y, z, t) \tag{10}$$

describes time evolution of temperature in heat conducting materials.

3. **The Laplace equation**

$$\nabla^2 u = 0, \quad u = u(x, y, z) \tag{11}$$

describes electric potential and also stationary distribution of temperature in a heat conducting material.

4. **The Schrödiger equation**

$$-\frac{\hbar}{2m} \nabla^2 \psi + V(x, y, z) \psi = i\hbar \psi_t, \quad \psi = \psi(x, y, z, t) \tag{12}$$

describes evolution of the wave function ψ in quantum mechanics.

Linear equations and the principle of superposition

A general PDE of second order in the variables $x_1, x_2, \ldots, x_n$ is given by

$$\sum_{i,j=1}^{n} A_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} B_i(x) u_{x_i} + F(x) u = G(x). \tag{13}$$

$$u_{x_i x_j} = u_{x_j x_i} \quad \Rightarrow \quad \text{we can assume that } A_{ij} = A_{ji} \tag{14}$$

To equation (13) we associate the differential operator

$$L = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i} + F. \tag{15}$$

Equation (13) in operator form:

$$L[u] = G. \tag{16}$$

- If $G = 0$, then we say that the equation is *homogeneous*.
- If $G \neq 0$, then we say that the equation is *inhomogeneous*.

Linear equations and the principle of superposition

A general PDE of second order in the variables $x_1, x_2, \ldots, x_n$ is given by

$$\sum_{i,j=1}^{n} A_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} B_i(x) u_{x_i} + F(x) u = G(x). \tag{13}$$

$$u_{x_i x_j} = u_{x_j x_i} \quad \Rightarrow \quad \text{we can assume that } A_{ij} = A_{ji} \tag{14}$$

To equation (13) we associate the differential operator

$$L = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i} + F. \tag{15}$$

Equation (13) in operator form:

$$L[u] = G. \tag{16}$$

- If $G = 0$, then we say that the equation is *homogeneous*.
- If $G \neq 0$, then we say that the equation is *inhomogeneous*.

Linear equations and the principle of superposition

A general PDE of second order in the variables $x_1, x_2, \ldots, x_n$ is given by

$$\sum_{i,j=1}^{n} A_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} B_i(x) u_{x_i} + F(x)u = G(x). \tag{13}$$

$$u_{x_i x_j} = u_{x_j x_i} \quad \Rightarrow \quad \text{we can assume that } A_{ij} = A_{ji} \tag{14}$$

To equation (13) we associate the differential operator

$$L = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i} + F. \tag{15}$$

Equation (13) in operator form:

$$L[u] = G. \tag{16}$$

- If $G = 0$, then we say that the equation is *homogeneous*.
- If $G \neq 0$, then we say that the equation is *inhomogeneous*.

Linear equations and the principle of superposition

A general PDE of second order in the variables $x_1, x_2, \dots, x_n$ is given by

$$\sum_{i,j=1}^{n} A_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} B_i(x)u_{x_i} + F(x)u = G(x). \tag{13}$$

$$u_{x_i x_j} = u_{x_j x_i} \quad \Rightarrow \quad \text{we can assume that } A_{ij} = A_{ji} \tag{14}$$

To equation (13) we associate the differential operator

$$L = \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i} + F. \tag{15}$$

Equation (13) in operator form:

$$L[u] = G. \tag{16}$$

- If $G = 0$, then we say that the equation is *homogeneous*.
- If $G \neq 0$, then we say that the equation is *inhomogeneous*.

A differential operator L is linear if

$$L[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 L[u_1] + \alpha_2 L[u_2], \quad \alpha_1, \alpha_2 \in \mathbb{R}. \tag{17}$$

Linear operators satisfy the *principle of superposition*. If

$$L[u_1] = 0, \quad L[u_2] = 0, \tag{18}$$

then the fuction $u = \alpha_1 u_1 + \alpha_2 u_2$ is a solution of the same equation because

$$L[u] = \alpha_1 L[u_1] + \alpha_2 L[u_2] = 0. \tag{19}$$

Elementary techniques

1. Direct integration
2. The method of substitution – introduction of new variables
3. Reduction of the number of variables using symmetry properties of the equation

Examples

1. Determine the general solution of the equation $u_{yy} = 2$ for the function $u = u(x, y)$.
2. Find the general solution of the equation $u_x - u_y = 0$.
3. Determine the general solution of the equation $u_{xy} + \frac{1}{x} u_y = \frac{y}{x^2}$.
4. Determine spherically symmetric solution of the Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

5.
   - Show that $u(x, y) = xf(2x + y)$ is the general solution of the equation $xu_x - 2xu_y = u$.
   - Determine the solution satisfying the condition $u(1, y) = y^2$.

Elementary techniques

1. Direct integration
2. The method of substitution – introduction of new variables
3. Reduction of the number of variables using symmetry properties of the equation

Examples

1. Determine the general solution of the equation $u_{yy} = 2$ for the function $u = u(x, y)$.
2. Find the general solution of the equation $u_x - u_y = 0$.
3. Determine the general solution of the equation $u_{xy} + \frac{1}{x} u_y = \frac{y}{x^2}$.
4. Determine spherically symmetric solution of the Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

5.
   - Show that $u(x, y) = xf(2x + y)$ is the general solution of the equation $xu_x - 2xu_y = u$.
   - Determine the solution satisfying the condition $u(1, y) = y^2$.

1. Direct integration
2. The method of substitution – introduction of new variables
3. Reduction of the number of variables using symmetry properties of the equation

Examples

1. Determine the general solution of the equation $u_{yy} = 2$ for the function $u = u(x, y)$.
2. Find the general solution of the equation $u_x - u_y = 0$.
3. Determine the general solution of the equation $u_{xy} + \frac{1}{x} u_y = \frac{y}{x^2}$.
4. Determine spherically symmetric solution of the Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

5.
   - Show that $u(x, y) = xf(2x + y)$ is the general solution of the equation $xu_x - 2xu_y = u$.
   - Determine the solution satisfying the condition $u(1, y) = y^2$.

1. Direct integration
2. The method of substitution – introduction of new variables
3. Reduction of the number of variables using symmetry properties of the equation

Examples

1. Determine the general solution of the equation $u_{yy} = 2$ for the function $u = u(x, y)$.
2. Find the general solution of the equation $u_x - u_y = 0$.
3. Determine the general solution of the equation $u_{xy} + \frac{1}{x} u_y = \frac{y}{x^2}$.
4. Determine spherically symmetric solution of the Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

5.
   - Show that $u(x, y) = xf(2x + y)$ is the general solution of the equation $xu_x - 2xu_y = u$.
   - Determine the solution satisfying the condition $u(1, y) = y^2$.

Elementary techniques

1. Direct integration
2. The method of substitution – introduction of new variables
3. Reduction of the number of variables using symmetry properties of the equation

Examples

1. Determine the general solution of the equation $u_{yy} = 2$ for the function $u = u(x, y)$.
2. Find the general solution of the equation $u_x - u_y = 0$.
3. Determine the general solution of the equation $u_{xy} + \frac{1}{x} u_y = \frac{y}{x^2}$.
4. Determine spherically symmetric solution of the Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

5.
   - Show that $u(x, y) = xf(2x + y)$ is the general solution of the equation $xu_x - 2xu_y = u$.
   - Determine the solution satisfying the condition $u(1, y) = y^2$.

Elementary techniques

1. Direct integration
2. The method of substitution – introduction of new variables
3. Reduction of the number of variables using symmetry properties of the equation

Examples

1. Determine the general solution of the equation $u_{yy} = 2$ for the function $u = u(x, y)$.
2. Find the general solution of the equation $u_x - u_y = 0$.
3. Determine the general solution of the equation $u_{xy} + \frac{1}{x} u_y = \frac{y}{x^2}$.
4. Determine spherically symmetric solution of the Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

5.
   - Show that $u(x, y) = xf(2x + y)$ is the general solution of the equation $xu_x - 2xu_y = u$.
   - Determine the solution satisfying the condition $u(1, y) = y^2$.

Initial and boundary conditions

Partial differential equations generally have infinitely many solutions. In order to determine a unique solution we have to impose **intial** and/or **boundary** conditions. These conditions naturally arise from the physical problem under consideration.

Example 1 The wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L,\ t > 0. \tag{20}$$

$u(x,t)$ wave amplitude at the point x at the moment t.

Initial conditions:

$$u(x,0) = f(x), \quad x \in [0,L], \tag{21}$$

$$u_t(x,0) = g(x), \quad x \in [0,L]. \tag{22}$$

Dirichlet boundary condition:

$$u(0,t) = u(L,t) = 0, \quad t \geq 0. \tag{23}$$

Neumann boundary condition:

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geq 0. \tag{24}$$

Initial and boundary conditions

Partial differential equations generally have infinitely many solutions. In order to determine a unique solution we have to impose **intial** and/or **boundary** conditions. These conditions naturally arise from the physical problem under consideration.

Example 1 The wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \ t > 0. \tag{20}$$

$u(x,t)$ wave amplitude at the point x at the moment t.

Initial conditions:

$$u(x,0) = f(x), \quad x \in [0,L], \tag{21}$$

$$u_t(x,0) = g(x), \quad x \in [0,L]. \tag{22}$$

Dirichlet boundary condition:

$$u(0,t) = u(L,t) = 0, \quad t \geq 0. \tag{23}$$

Neumann boundary condition:

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geq 0. \tag{24}$$

Initial and boundary conditions

Partial differential equations generally have infinitely many solutions. In order to determine a unique solution we have to impose **intial** and/or **boundary** conditions. These conditions naturally arise from the physical problem under consideration.

Example 1 The wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \ t > 0. \tag{20}$$

$u(x,t)$ wave amplitude at the point x at the moment t.

Initial conditions:

$$u(x,0) = f(x), \quad x \in [0,L], \tag{21}$$

$$u_t(x,0) = g(x), \quad x \in [0,L]. \tag{22}$$

Dirichlet boundary condition:

$$u(0,t) = u(L,t) = 0, \quad t \geq 0. \tag{23}$$

Neumann boundary condition:

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geq 0. \tag{24}$$

Initial and boundary conditions

Partial differential equations generally have infinitely many solutions. In order to determine a unique solution we have to impose **intial** and/or **boundary** conditions. These conditions naturally arise from the physical problem under consideration.

Example 1 The wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \ t > 0. \tag{20}$$

$u(x,t)$ wave amplitude at the point x at the moment t.

Initial conditions:

$$u(x,0) = f(x), \quad x \in [0, L], \tag{21}$$

$$u_t(x,0) = g(x), \quad x \in [0, L]. \tag{22}$$

Dirichlet boundary condition:

$$u(0,t) = u(L,t) = 0, \quad t \geq 0. \tag{23}$$

Neumann boundary condition:

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geq 0. \tag{24}$$

Initial and boundary conditions

Partial differential equations generally have infinitely many solutions. In order to determine a unique solution we have to impose **intial** and/or **boundary** conditions. These conditions naturally arise from the physical problem under consideration.

Example 1 The wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L,\ t > 0. \tag{20}$$

$u(x,t)$ wave amplitude at the point x at the moment t.

Initial conditions:

$$u(x,0) = f(x), \quad x \in [0,L], \tag{21}$$

$$u_t(x,0) = g(x), \quad x \in [0,L]. \tag{22}$$

Dirichlet boundary condition:

$$u(0,t) = u(L,t) = 0, \quad t \geq 0. \tag{23}$$

Neumann boundary condition:

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geq 0. \tag{24}$$

Example 2 The Laplace equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega, \tag{25}$$

$\Omega \subseteq \mathbb{R}^2$ domain bounded by a closed piecewise smooth curve $\partial\Omega$.

Dirichlet boundary condition:

$$u(x, y) = g(x, y), \quad (x, y) \in \partial\Omega \tag{26}$$

Neumann boundary condition:

$$\frac{\partial u}{\partial \vec{n}}(x, y) = g(x, y), \quad (x, y) \in \partial\Omega, \tag{27}$$

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} \quad \text{normal derivative to the curve} \quad \partial\Omega \tag{28}$$

Example 2 The Laplace equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega, \tag{25}$$

$\Omega \subseteq \mathbb{R}^2$ domain bounded by a closed piecewise smooth curve $\partial\Omega$.

Dirichlet boundary condition:

$$u(x, y) = g(x, y), \quad (x, y) \in \partial\Omega \tag{26}$$

Neumann boundary condition:

$$\frac{\partial u}{\partial \vec{n}}(x, y) = g(x, y), \quad (x, y) \in \partial\Omega, \tag{27}$$

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} \quad \text{normal derivative to the curve} \quad \partial\Omega \tag{28}$$

Example 2 The Laplace equation

$$u_{xx} + u_{yy} = 0, \quad (x,y) \in \Omega, \tag{25}$$

$\Omega \subseteq \mathbb{R}^2$ domain bounded by a closed piecewise smooth curve $\partial\Omega$.

Dirichlet boundary condition:

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega \tag{26}$$

Neumann boundary condition:

$$\frac{\partial u}{\partial \vec{n}}(x,y) = g(x,y), \quad (x,y) \in \partial\Omega, \tag{27}$$

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} \quad \text{normal derivative to the curve} \quad \partial\Omega \tag{28}$$

Stability of solutions

We say that a partial differential equation with given intial and boundary conditions is a well posed problem if it satisfies the following conditions (J. Hadamard, 1902.):

- *existence:* the problem has a solution,
- *uniqueness:* the solution of the problem is unique for given initial and boundary conditions,
- *stability:* the solution continously depends on the equation parameters and given intial and boundary conditions.

A partial differential equation is stable if small perturbations of initial and boundary conditions or equation parameters cause small changes of the solution.

Stability of solutions

We say that a partial differential equation with given intial and boundary conditions is a well posed problem if it satisfies the following conditions (J. Hadamard, 1902.):

- *existence:* the problem has a solution,
- *uniqueness:* the solution of the problem is unique for given initial and boundary conditions,
- *stability:* the solution continously depends on the equation parameters and given intial and boundary conditions.

A partial differential equation is stable if small perturbations of initial and boundary conditions or equation parameters cause small changes of the solution.

Stability of solutions

We say that a partial differential equation with given intial and boundary conditions is a well posed problem if it satisfies the following conditions (J. Hadamard, 1902.):

- *existence:* the problem has a solution,
- *uniqueness:* the solution of the problem is unique for given initial and boundary conditions,
- *stability:* the solution continously depends on the equation parameters and given intial and boundary conditions.

A partial differential equation is stable if small perturbations of initial and boundary conditions or equation parameters cause small changes of the solution.

Stability of solutions

We say that a partial differential equation with given intial and boundary conditions is a well posed problem if it satisfies the following conditions (J. Hadamard, 1902.):

- *existence:* the problem has a solution,
- *uniqueness:* the solution of the problem is unique for given initial and boundary conditions,
- *stability:* the solution continously depends on the equation parameters and given intial and boundary conditions.

A partial differential equation is stable if small perturbations of initial and boundary conditions or equation parameters cause small changes of the solution.

Stability of solutions

We say that a partial differential equation with given intial and boundary conditions is a well posed problem if it satisfies the following conditions (J. Hadamard, 1902.):

- *existence:* the problem has a solution,
- *uniqueness:* the solution of the problem is unique for given initial and boundary conditions,
- *stability:* the solution continously depends on the equation parameters and given intial and boundary conditions.

A partial differential equation is stable if small perturbations of initial and boundary conditions or equation parameters cause small changes of the solution.

Stability of solutions

We say that a partial differential equation with given intial and boundary conditions is a well posed problem if it satisfies the following conditions (J. Hadamard, 1902.):

- *existence:* the problem has a solution,
- *uniqueness:* the solution of the problem is unique for given initial and boundary conditions,
- *stability:* the solution continously depends on the equation parameters and given intial and boundary conditions.

A partial differential equation is stable if small perturbations of initial and boundary conditions or equation parameters cause small changes of the solution.

Example

1. Formulate the condition of stability for the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega, \tag{29}$$

   with respect to the Dirichlet boundary condition.

2. Formulate the condition of stability of the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \tag{30}$$

   in the time interval $[0, T]$ with respect to the initial conditions.

Example

1. Formulate the condition of stability for the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega, \tag{29}$$

with respect to the Dirichlet boundary condition.

2. Formulate the condition of stability of the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \tag{30}$$

in the time interval $[0, T]$ with respect to the initial conditions.

The following examples illustrate problem that are not well posed.

Hadamard's example

Show that the solution of the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2 \tag{31}$$

with boundary condions on the line $y = 0$,

$$u(x, 0) = f(x), \quad u_y(x, 0) = g(x), \tag{32}$$

is not stable.

Backwards heat equation

Show that the solution of the equation

$$u_t + u_{xx} = 0, \quad -\infty < x < \infty, \ t > 0, \tag{33}$$

with initial condition $u(x, 0) = 1$ is not stable.

The following examples illustrate problem that are not well posed.

Hadamard's example

Show that the solution of the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2 \tag{31}$$

with boundary condions on the line $y = 0$,

$$u(x, 0) = f(x), \quad u_y(x, 0) = g(x), \tag{32}$$

is not stable.

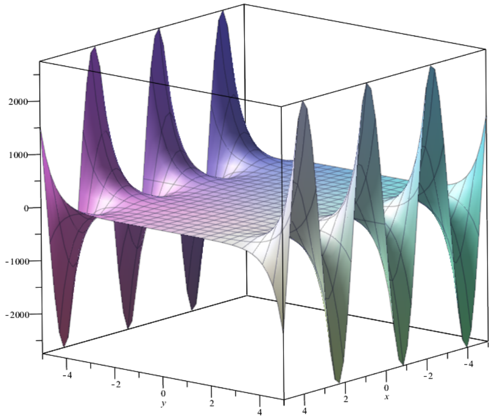
Backwards heat equation

Show that the solution of the equation

$$u_t + u_{xx} = 0, \quad -\infty < x < \infty,\ t > 0, \tag{33}$$

with initial condition $u(x, 0) = 1$ is not stable.

Instabitliy of solution in Hadamard's example



The theory of Fourier series arouse as a method for solving the heat equation by the method of separation of variables.

Problem: Can a given function $f\colon [-L, L] \to \mathbb{R}$ be written as a linear combination of the functions $\sin(\omega x)$ i $\cos(\omega x)$ with different periods?

Under certain conditions on f, it is possible to write f as a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad -L \leq x \leq L. \qquad (34)$$

The Fourier series is a linear combination of oscillatory functions with periods and frequencies given by

$$T_n = \frac{2L}{n}, \quad f_n = \frac{1}{T_n} = \frac{n}{2L}. \qquad (35)$$

The coefficients a_n i b_n determine the "amount" of each frequency present in the fuction.

The theory of Fourier series arouse as a method for solving the heat equation by the method of separation of variables.

Problem: Can a given function $f\colon [-L, L] \to \mathbb{R}$ be written as a linear combination of the functions $\sin(\omega x)$ i $\cos(\omega x)$ with different periods?

Under certain conditions on f, it is possible to write f as a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad -L \leq x \leq L. \tag{34}$$

The Fourier series is a linear combination of oscillatory functions with periods and frequencies given by

$$T_n = \frac{2L}{n}, \quad f_n = \frac{1}{T_n} = \frac{n}{2L}. \tag{35}$$

The coefficients a_n i b_n determine the "amount" of each frequency present in the fuction.

Trigonometric functions $\sin(xn)$, $n = 1, 2, 3, \ldots$

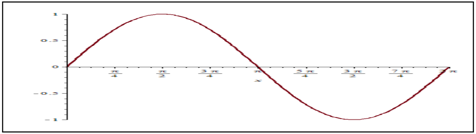


Figure: $f(x) = \sin(x)$

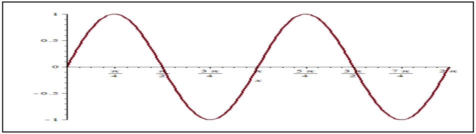


Figure: $f(x) = \sin(2x)$

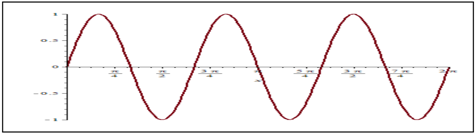


Figure: $f(x) = \sin(3x)$

For the moment, let us suppose that it is possible to write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad -L \leq x \leq L. \qquad (36)$$

Problem: How can we determine the coefficients a_n i b_n?

On the space of continous functions $C([-L,L])$ we can define the inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)dx. \qquad (37)$$

The set

$$\left\{ 1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \mid n \in \mathbb{N} \right\} \qquad (38)$$

is orthogonal with respect to the inner product (37).

For the moment, let us suppose that it is possible to write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad -L \leq x \leq L. \tag{36}$$

Problem: How can we determine the coefficients a_n i b_n?

On the space of continous functions $C([-L, L])$ we can define the inner product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)dx. \tag{37}$$

The set

$$\left\{ 1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right) \mid n \in \mathbb{N} \right\} \tag{38}$$

is orthogonal with respect to the inner product (37).

The Fourier coefficients can be determined by using the orthogonality relations:

$$\int_{-L}^{L} 1 \cdot \cos\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} 1 \cdot \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad n \geq 1, \tag{39}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\,\delta_{nm}, \tag{40}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\,\delta_{nm}, \tag{41}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad n, m \geq 1. \tag{42}$$

Kronecker delta symbol

$$\delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \tag{43}$$

Assuming that one can interchange summation and integration, determine the expressions for the Fourier coefficients a_n and b_n.

The Fourier coefficients can be determined by using the orthogonality relations:

$$\int_{-L}^{L} 1 \cdot \cos\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L} 1 \cdot \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad n \geq 1, \tag{39}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\,\delta_{nm}, \tag{40}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\,\delta_{nm}, \tag{41}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad n, m \geq 1. \tag{42}$$

Kronecker delta symbol

$$\delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases} \tag{43}$$

Assuming that one can interchange summation and integration, determine the expressions for the Fourier coefficients a_n and b_n.

Definition

The trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{44}$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \ldots \tag{45}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \ldots \tag{46}$$

is called the *Fourier series* of the fuction f on the interval $[-L, L]$. The coefficients a_n and b_n are called the *Fourier coefficients* of the function f.

What can we say about the convergence of the Fourier series?

1. The Fourier coefficients are completely determined by the integral of f, hence the Fourier series of f remains unchanged if the function is the values of f are changed at countably many points. This implies that the Fourier series need not converge to $f(x)$ at evey point $x \in [-L, L]$.
2. The convergence of the Fourier series can be pointwise, uniform or in L^2–norm, depending on the properties of the function f.

Problem

We want to determine conditions on the function f such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad -L \le x \le L. \tag{47}$$

What can we say about the convergence of the Fourier series?

1. The Fourier coefficients are completely determined by the integral of f, hence the Fourier series of f remains unchanged if the function is the values of f are changed at countably many points. This implies that the Fourier series need not converge to $f(x)$ at evey point $x \in [-L, L]$.
2. The convergence of the Fourier series can be pointwise, uniform or in L^2–norm, depending on the properties of the function f.

Problem

We want to determine conditions on the function f such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad -L \leq x \leq L. \qquad (47)$$

What can we say about the convergence of the Fourier series?

1. The Fourier coefficients are completely determined by the integral of f, hence the Fourier series of f remains unchanged if the function is the values of f are changed at countably many points. This implies that the Fourier series need not converge to $f(x)$ at evey point $x \in [-L, L]$.
2. The convergence of the Fourier series can be pointwise, uniform or in L^2–norm, depending on the properties of the function f.

Problem

We want to determine conditions on the function f such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \quad -L \leq x \leq L. \tag{47}$$

Elementary examples

When computing the Fourier coefficients it is useful to remember the following rules:

1. if $h\colon [-L, L] \to \mathbb{R}$ is an odd function, then

$$\int_{-L}^{L} h(x)dx = 0, \tag{48}$$

2. if $h\colon [-L, L] \to \mathbb{R}$ is an even function, then

$$\int_{-L}^{L} h(x)dx = 2\int_{0}^{L} h(x)dx. \tag{49}$$

Examples

1. Determine the Fourier series of $f(x) = x$ in the interval $[-L, L]$.
2. Find the Fourier series of $f(x) = x^2 - 1$ in the interval $[-1, 1]$.

Elementary examples

When computing the Fourier coefficients it is useful to remember the following rules:

1. if $h\colon [-L, L] \to \mathbb{R}$ is an odd function, then

$$\int_{-L}^{L} h(x)dx = 0, \tag{48}$$

2. if $h\colon [-L, L] \to \mathbb{R}$ is an even function, then

$$\int_{-L}^{L} h(x)dx = 2\int_{0}^{L} h(x)dx. \tag{49}$$

Examples

1. Determine the Fourier series of $f(x) = x$ in the interval $[-L, L]$.
2. Find the Fourier series of $f(x) = x^2 - 1$ in the interval $[-1, 1]$.

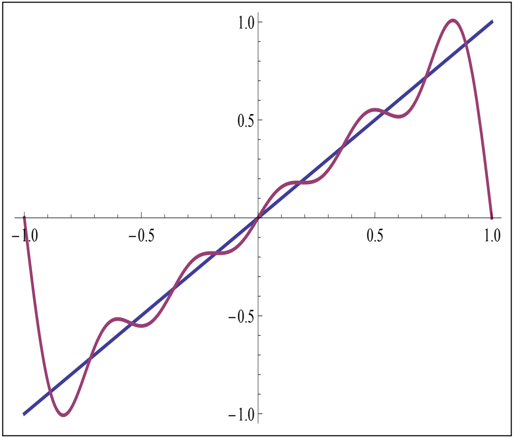


Figure: The Fourier series of $f(x) = x$, $N = 5$.

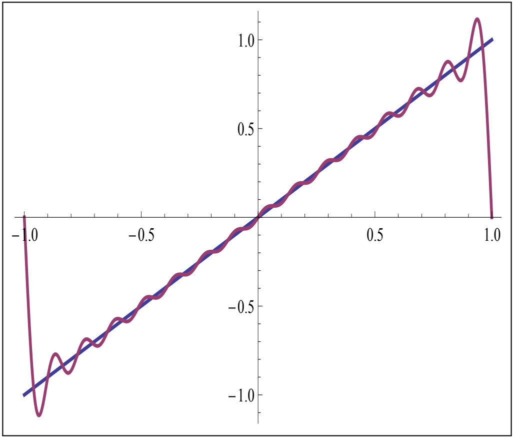


Figure: The Fourie series of $f(x) = x$, $N = 15$.

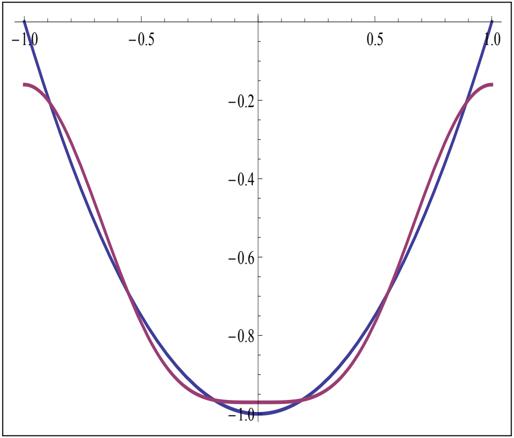


Figure: The Fourier series of $f(x) = x^2 - 1$, $N = 2$.

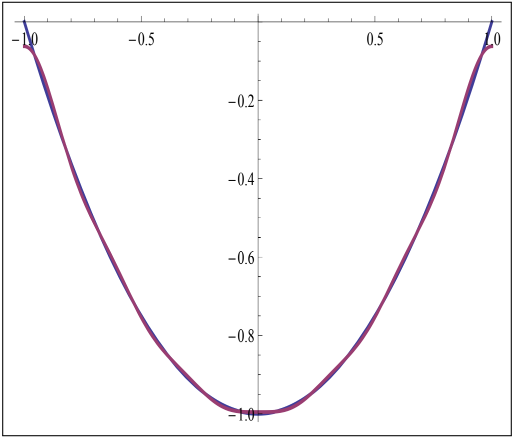


Figure: The Fourier series of $f(x) = x^2 - 1$, $N = 6$.

Definicija

A function f is piecewise continuous on the interval $[a, b]$ if

1. it is defined and continuous on $[a, b]$ except possibly at finitely many points $a \leq x_1 < x_2 < \ldots < x_n \leq b$,
2. at the points $x_k \neq a, b$, one–sided limits exist

$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \quad f(x_k^+) = \lim_{x \to x_k^+} f(x), \tag{50}$$

3. at the boundary points there exist limits $\lim_{x \to a^+} f(x)$ i $\lim_{x \to b^-} f(x)$.

Such functions have a jump discontinuity at x_k and a the change of value

$$\beta_k = f(x_k^+) - f(x_k^-). \tag{51}$$

Definicija

A function f is piecewise C^1 on the interval $[a, b]$ if f and f' are piecewise continuous on $[a, b]$.

Example

Determine if the functions f and g defined by

$$f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 2, & x = 0, \\ x^2, & 0 < x \leq 1, \end{cases} \tag{52}$$

$$g(x) = |x| \tag{53}$$

are piecewise C^1 on $[-1, 1]$.

Definicija

A function f is piecewise C^1 on the interval $[a, b]$ if f and f' are piecewise continuous on $[a, b]$.

Example

Determine if the functions f and g defined by

$$f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 2, & x = 0, \\ x^2, & 0 < x \leq 1, \end{cases} \tag{52}$$

$$g(x) = |x| \tag{53}$$

are piecewise C^1 on $[-1, 1]$.

Dirichlet theorem

Suppose f is a piecewise C^1 function on $[-L, L]$ and let $\tilde{f}$ be the Fourier series of f. Then

1. $\tilde{f}(x_0) = f(x_0)$ if f is continuous at the point $x_0 \in (-L, L)$,
2. $\tilde{f}(x_0) = \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$ if f has a discontinuity at $x_0 \in (-L, L)$,
3. $\tilde{f}(\pm L) = \frac{1}{2}\left[f(-L^+) + f(L^-)\right]$.

Remarks:

1. Continuity of the function is not a sufficient condition for convergence of its Fourier series.
2. If f is continuous on $[-L, L]$, then its derivative must be piecewise continuous on $[-L, L]$ in order to have converence of $\tilde{f}(x)$ to $f(x)$ at every point $x \in (-L, L)$.

Dirichlet theorem

Suppose f is a piecewise C^1 function on $[-L, L]$ and let $\tilde{f}$ be the Fourier series of f. Then

1. $\tilde{f}(x_0) = f(x_0)$ if f is continuous at the point $x_0 \in (-L, L)$,
2. $\tilde{f}(x_0) = \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right]$ if f has a discontinuity at $x_0 \in (-L, L)$,
3. $\tilde{f}(\pm L) = \frac{1}{2}\left[f(-L^+) + f(L^-)\right]$.

Remarks:

1. Continuity of the function is not a sufficient condition for convergence of its Fourier series.
2. If f is continuous on $[-L, L]$, then its derivative must be piecewise continuous on $[-L, L]$ in order to have converence of $\tilde{f}(x)$ to $f(x)$ at every point $x \in (-L, L)$.

Example

Let us illustrate the Dirichlet theorem with the functions

$$f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ 1, & 0 \leq x \leq 1, \end{cases} \tag{54}$$

$$g(x) = |x|, \quad x \in [-\pi, \pi]. \tag{55}$$

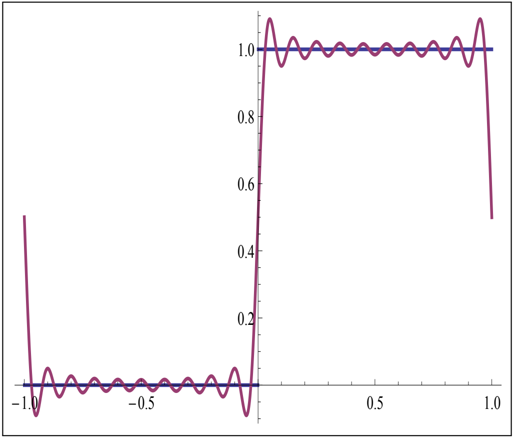


Figure: The Fourier series of the step function.

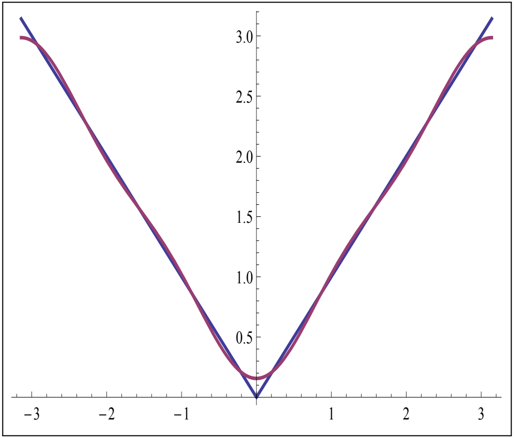


Figure: The Fourier series of $f(x) = |x|$.

Uniform convergence

Cauchy–Schwartz inequality

If z_i, w_i are complex numbers, $1 \le i \le n$, then

$$\left| \sum_{i=1}^{n} z_i \bar{w}_i \right| \le \sqrt{\sum_{i=1}^{n} |z_i|^2} \sqrt{\sum_{i=1}^{n} |w_i|^2}. \tag{56}$$

Bessel inequality

Suppose $f : [-L, L] \to \mathbb{R}$ is a picewise continuous function and let

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \ge 0, \tag{57}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \ge 1. \tag{58}$$

Then

$$\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \le \frac{1}{L} \int_{-L}^{L} f^2(x) dx. \tag{59}$$

Uniform convergence

Cauchy–Schwartz inequality

If z_i, w_i are complex numbers, $1 \leq i \leq n$, then

$$\left|\sum_{i=1}^{n} z_i \bar{w}_i\right| \leq \sqrt{\sum_{i=1}^{n} |z_i|^2}\sqrt{\sum_{i=1}^{n} |w_i|^2}. \tag{56}$$

Bessel inequality

Suppose $f\colon [-L, L] \to \mathbb{R}$ is a picewise continuous function and let

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0, \tag{57}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1. \tag{58}$$

Then

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{L}\int_{-L}^{L} f^2(x)dx. \tag{59}$$

A consequence of Bessel inequality is

Riemann-Lebesgue lemma

If $f\colon [-L,L] \to \mathbb{R}$ is a piecewise continuous function, then

$$\lim_{n\to\infty} a_n = \frac{1}{L}\lim_{n\to\infty}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) = 0, \tag{60}$$

$$\lim_{n\to\infty} b_n = \frac{1}{L}\lim_{n\to\infty}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) = 0. \tag{61}$$

One can show that a stronger result holds.

Parseval equality

If $f\colon [-L,L] \to \mathbb{R}$ is a piecewise continuous function, then

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} |f(x)|^2 dx. \tag{62}$$

A consequence of Bessel inequality is

Riemann-Lebesgue lemma

If $f\colon [-L, L] \to \mathbb{R}$ is a piecewise continuous function, then

$$\lim_{n\to\infty} a_n = \frac{1}{L} \lim_{n\to\infty} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) = 0, \quad (60)$$

$$\lim_{n\to\infty} b_n = \frac{1}{L} \lim_{n\to\infty} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) = 0. \quad (61)$$

One can show that a stronger result holds.

Parseval equality

If $f\colon [-L, L] \to \mathbb{R}$ is a piecewise continuous function, then

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{L} \int_{-L}^{L} |f(x)|^2 dx. \quad (62)$$

Uniform convergence of Fourier series

Suppose f is a continuous and piecewise C^1 function on the inverval $[-L, L]$ and $f(-L) = f(L)$. Then the Fourier series converges uniformly to f on $[-L, L]$.

If the Fourier coefficients are not known explicitly, one can derive the following bounds for $|a_n|$ and $|b_n|$.

Teorem

Suppose $f \in C^2[-L, L]$ such that $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Let $M = \max_{x\in[-L,L]} |f''(x)|$. Then the Fourier coefficienta are bounded by

$$|a_n| = \left| \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \right| \leq \frac{2L^2 M}{\pi^2 n^2}, \quad (63)$$

$$|b_n| = \left| \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right| \leq \frac{2L^2 M}{\pi^2 n^2}, \quad n \geq 1. \quad (64)$$

Uniform convergence of Fourier series

Suppose f is a continuous and piecewise C^1 function on the inverval $[-L, L]$ and $f(-L) = f(L)$. Then the Fourier series converges uniformly to f on $[-L, L]$.

If the Fourier coefficients are not known explicitly, one can derive the following bounds for $|a_n|$ and $|b_n|$.

Teorem

Suppose $f \in C^2[-L, L]$ such that $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Let $M = \max_{x\in[-L,L]} |f''(x)|$. Then the Fourier coefficienta are bounded by

$$|a_n| = \left| \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \right| \leq \frac{2L^2 M}{\pi^2 n^2}, \tag{63}$$

$$|b_n| = \left| \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right| \leq \frac{2L^2 M}{\pi^2 n^2}, \quad n \geq 1. \tag{64}$$

Using the above theorem we can estimate the number of Fourier coefficients needed to approximate a function within a given accuracy.

Let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] \tag{65}$$

be the N–th partial sum of the Fourier series of f. If we require that

$$\sup_{x\in[-L,L]} |f(x) - S_N(x)| < \varepsilon, \tag{66}$$

then we need at least

$$N > \frac{4L^2M}{\pi^2\epsilon} \tag{67}$$

Fouerier coefficients.