Partial differential equations

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PMF–Split

Second order equations

A general second order PDE in two independent variables is given by

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G. \quad (1)$$

The functions u, A, B, C, D, E, F, G depend on the variables $(x, y) \in \Omega \subseteq \mathbb{R}^2$.

Operator form of the equation:

$$L[u] = G, \quad L = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F \quad (2)$$

The principal part of L:

$$L_0 = A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x \partial y} + C\frac{\partial^2}{\partial y^2}. \quad (3)$$

To each equation (1) we associate the discriminant $\Delta(x, y)$:

$$\Delta(x, y) = B^2(x, y) - A(x, y)C(x, y). \quad (4)$$

Qualitative properties of solutons of Eq. (1) depend on the sign of $\Delta(x, y)$.

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Definition

Second order equation $L[u] = G$ is called

1. hyperbolic at the point (x, y) if $\Delta(x, y) > 0$,
2. parabolic at the point (x, y) if $\Delta(x, y) = 0$,
3. elliptic at the point (x, y) if $\Delta(x, y) < 0$.

If the equation $L[u] = G$ is hyperbolic (parabolic, elliptic) at every point in the domain Ω, then we say that the equation is hyperbolic (parabolic, elliptic) in Ω.

Examples

Classify the following equations:

1. $u_{tt} - c^2 u_{xx} = 0$,
2. $u_t - k u_{xx} = 0$,
3. $u_{xx} + u_{yy} = 0$,
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The type of equation is invariant with respect to a regular transformation of variables in the equation.

Lemma

Consider a second order equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G. \tag{5}$$

If $\alpha = \alpha(x,y),\ \beta = \beta(x,y)$ is a regular transformation of variables, then the sign of the discriminant $\Delta = B^2 - AC$ is invariant with respect to the transformation $(x,y) \mapsto (\alpha, \beta)$.

By introducing new variables, every equation of second order can be transformed into a *canonical form*.

Transformed coefficients in new variables:

$$\bar{A} = A\alpha_x^2 + 2B\alpha_x\alpha_y + C\alpha_y^2, \tag{6}$$

$$\bar{B} = A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + C\alpha_y\beta_y, \tag{7}$$

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Canonical forms

Definition

1. The canonical form of a **hyperbolic** equation is
$$u_{xy} + L_1[u] = G \tag{9}$$
where L_1 is a differential operator of first order. This canonical form is equivalent with
$$w_{\alpha\alpha} - w_{\beta\beta} + L_1[w] = G \tag{10}$$
where the variables α, β are given by $\alpha = x + y$, $\beta = x - y$.
2. The canonical form of a **parabolic** equation is
$$u_{xx} + L_1[u] = G. \tag{11}$$
3. The canonical form an **elliptic** equation is
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Theorem

Suppose

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{13}$$

is a hyperbolic equation in the domain $\Omega \subseteq \mathbb{R}^2$. Then there exist variables $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ in which the equation (13) has the canonical form

$$w_{\alpha\beta} + L_1[w] = \bar{G} \tag{14}$$

where $w(\alpha, \beta) = u(x(\alpha, \beta), y(\alpha, \beta))$ and L_1 is a differential operator of first order.

Example

Determine the canonical form and find a general solution of the equation

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2. \tag{15}$$

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is a parabolic equation in the domain $\Omega \subseteq \mathbb{R}^2$. Then there exist variables $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ in which the equation (16) has the canonical form

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$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} + xu_x + yu_y = 0 \tag{18}$$

in the half–plane $x > 0$.

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is an elliptic equation in the region $\Omega \subseteq \mathbb{R}^2$. Then there exist variables $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ in which the equation (19) has the canonical form

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The heat equation

The heat equation

$$u_t - ku_{xx} = 0, \quad 0 < x < L,\ t > 0. \tag{22}$$

describes the temperature $u(x,t)$ in a thin heat conducing rod. The rod is insulated everywhere except possibly at the end points $x = 0$ i $x = L$.

Dirichlet boundary conditions:

$$u_t - ku_{xx} = 0, \qquad 0 < x < L,\ t > 0, \tag{23}$$

$$u(x,0) = f(x), \qquad 0 \leq x \leq L, \tag{24}$$

$$u(0,t) = a(t), \quad u(L,t) = b(t), \quad t \geq 0. \tag{25}$$

Neumann boundary conditions:

$$u_x(0,t) = a(t), \quad u_x(0,t) = b(t), \quad t \geq 0. \tag{26}$$

$u_x(x_0,t)$ describes heat flow at the point x_0

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Theorem (Uniqueness of solution)

If u_1 and u_2 are C^2 solutions of the problem

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then $u_1 = u_2$.

Definition

The parabolic border of the rectangle $D = [0,L] \times [0,T]$ is the union of its base and the vertical sides of the rectangle,

$$\partial_p D = \{(0,t) \mid 0 \le t \le T\} \cup \{(x,0) \mid 0 \le x \le L\} \cup \{(L,t) \mid 0 \le t \le T\}.$$

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The maximum principle

Suppose u is a C^2 solution of the heat equation

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \ t > 0. \tag{30}$$

Let $T > 0$ and let $D = [0, L] \times [0, T]$. Then

$$\max_{(x,t) \in D} u(x, t) = u(x_0, t_0) \tag{31}$$

at some point $(x_0, t_0) \in \partial_p D$.

The minimum principle

Every C^2 solution of the heat equation attains a minimum on the parabolic border of the rectangle $[0, L] \times [0, T]$.

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Theorem (Stability of solutions)

Suppose u_1 and u_2 C^2 are solutions of the initial–boundary value problems

$$\frac{\partial u_i}{\partial t} - k\frac{\partial^2 u_i}{\partial x^2} = 0, \quad 0 < x < L, \quad t > 0, \tag{32}$$

$$u_i(x,0) = f_i(x), \quad 0 \le x \le L, \tag{33}$$

$$u_i(0,t) = a_i(t), \quad u_i(L,t) = b_i(t), \quad t \ge 0 \tag{34}$$

for $i = 1, 2$. Let $T > 0$. If

$$\max_{0 \le x \le L} |f_1(x) - f_2(x)| < \varepsilon, \tag{35}$$

$$\max_{0 \le t \le T} |a_1(t) - a_2(t)| < \varepsilon, \quad \max_{0 \le t \le T} |b_1(t) - b_2(t)| < \varepsilon \tag{36}$$

for some $\varepsilon > 0$, then

$$\max_{(x,t) \in D} |u_1(x,t) - u_2(x,t)| < \varepsilon \tag{37}$$

where $D = [0, L] \times [0, T]$.

Separation of variables

Consider the heat equation with **Dirichlet** boundary condtions

$$u_t - ku_{xx} = 0, \qquad 0 < x < L, \quad t > 0, \tag{38}$$

$$u(x,0) = f(x), \qquad 0 \le x \le L, \tag{39}$$

$$u(0,t) = u(L,t) = 0, \quad t \ge 0. \tag{40}$$

Compatibility of the initial and boundary conditions: $f(0) = f(L) = 0$.

We seek solution in the separated form $u(x,t) = P(x)Q(t)$.

The solution is obtained in the form of an infinite series

$$u(x,t) = \sum_{n=1}^{n} P_n(x)Q_n(t). \tag{41}$$

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Theorem (Existence of solution)

Suppose the function $f\colon [0,L] \to \mathbb{R}$ satisfies the following conditions:

1. f is continuous and picewise C^1 on $[0,L]$,
2. $f(0) = f(L) = 0$.

Then the function

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right), \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{42}$$

is a classical solution of the initial–boundary value problem

$$u_t - ku_{xx} = 0, \qquad 0 < x < L, \quad t > 0, \tag{43}$$

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{44}$$

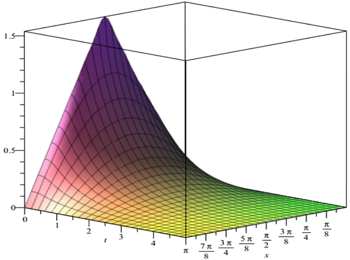
$$u(0,t) = u(L,t) = 0, \qquad t \ge 0. \tag{45}$$

Example Determine the solution of the heat equation

$$u_t - u_{xx} = 0, \quad 0 < x < \pi,\ t > 0 \tag{46}$$

$$u(0,t) = u(L,t) = 0, \tag{47}$$

$$u(x,0) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x \le \pi. \end{cases} \tag{48}$$



$$u(x,t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} e^{-(2m-1)^2 t} \sin\left((2m-1)x\right) \tag{49}$$

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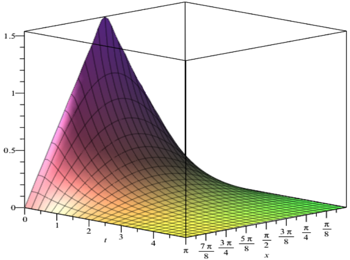

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Neumann boundary conditions

$$u_t - ku_{xx} = 0, \qquad 0 < x < L,\ t > 0, \tag{50}$$

$$u_x(0,t) = u_x(L,t) = 0, \quad t \geq 0, \tag{51}$$

$$u(x,0) = f(x), \qquad -L \leq x \leq L. \tag{52}$$

Compatibility of initial and boundary conditions: $f'(0) = 0$, $f'(L) = 0$.

Solution:

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-k(\frac{n\pi}{L})^2 t} \cos\left(\frac{n\pi}{L}x\right), \tag{53}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad n \geq 0. \tag{54}$$

Periodic boundary conditions

$$u_t - ku_{xx} = 0, \quad -L < x < L,\ t > 0, \tag{55}$$

$$u(x,0) = f(x), \quad -L \le x \le L, \tag{56}$$

$$u(-L,t) = u(L,t), \quad u_x(-L,t) = u_x(L,t), \quad t \ge 0. \tag{57}$$

Solution:

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-k(\frac{n\pi}{L})^2 t} \left[A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right)\right] \tag{58}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \quad n \ge 0, \tag{59}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx, \quad n \ge 1. \tag{60}$$

Nonhomogeneous heat equation

$$u_t - k u_{xx} = F(x,t), \qquad 0 < x < L,\ t > 0, \tag{61}$$

$$u(x,0) = f(x), \qquad 0 \le x \le L, \tag{62}$$

$$u(0,t) = u(L,t) = 0, \quad t \ge 0. \tag{63}$$

$F(x,t)$ models an internal heat source in the rod.

In the homogeneous case the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{n\pi}{L}x\right), \quad B_n(t) = B_n\, e^{-k\left(\frac{n\pi}{L}\right)^2 t}. \tag{64}$$

We seek solution by the method of variation of parameters:

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Example Solve the nonhomogeneous equation

$$u_t - u_{xx} = e^{-t}\sin(3x), \quad 0 < x < \pi,\ t > 0, \tag{66}$$

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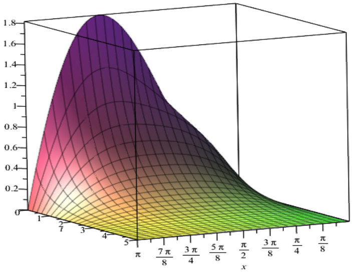
$$u(x,0) = x\sin(x), \qquad 0 \leq x \leq \pi. \tag{68}$$



$$u(x,t) = \frac{\pi}{2}e^{-t}\sin(x) + \frac{1}{8}(e^{-t} - e^{-9t})\sin(3x) - \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{n}{(4n^2-1)^2}e^{-4n^2t}\sin(2nx) \tag{69}$$

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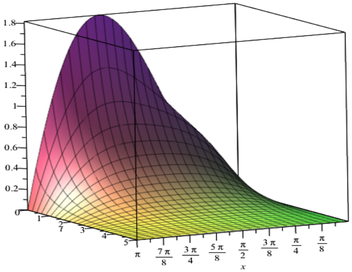

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The wave equation

The wave equation describes period motion such as oscillations in continuum mechanics or propagation of electromagnetic and sound waves.

The one–dimensional wave eqation

$$u_{tt} - c^2 u_{xx} = 0 \tag{70}$$

describes oscillations of an elastic wire under the following assumptions:

- dissipative effects (such as internal friction) are negligible,
- displacement $u(x,t)$ from the equilibrium position is perpendicular to the x–axis,
- no external forces act on the wire.

D'Alambert solution for homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{71}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}, \tag{72}$$

$f(x)$ initial amplitude, $g(x)$ initial velocity of the point x.

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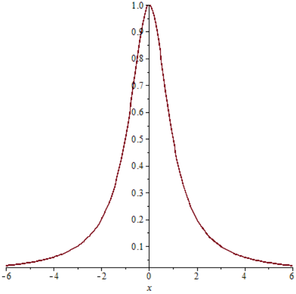
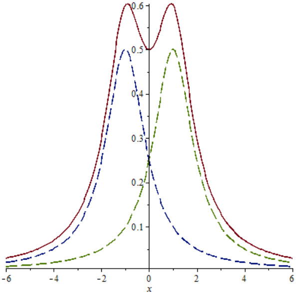
The general solution of the wave equation is a superposition of two traveling waves:

$$u(x,t) = A(x+ct) + B(x-ct). \tag{73}$$



(a) Initial profile $f(x)$



(b) Function $u(x,t)$ as a superposition of two traveling waves.

Slika:

Theorem

Let $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Then the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{74}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}, \tag{75}$$

has a unique solution

$$u(x,t) = \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \tag{76}$$

which is in every finite interval $0 \leq t \leq T$ stable with respect to the initial condtions (75).

D'Alambert soluton for nonhomogenous wave equation

$$u_{tt} - c^2 u_{xx} = F(x,t), \qquad x \in \mathbb{R}, \quad t > 0, \tag{77}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}. \tag{78}$$

$F(x,t)$ models an external force acting on the wire

Introduce the variable $y = ct$ and define $w(x,y) = u(x, y/c)$. Then w satisfies

$$w_{xx} - w_{yy} = F^*(x,y), \qquad F^*(x,y) = -\frac{1}{c^2}F(x,y), \tag{79}$$

$$w(x,0) = f(x), \tag{80}$$

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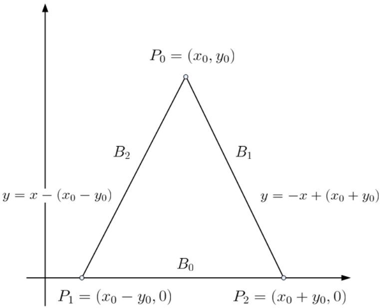
By applying the Green's theorem to the triangle in the picture, we can find the value of w at the point (x_0, y_0).



$$w(x_0, y_0) = \frac{1}{2}[f(x_0+y_0)+f(x_0-y_0)]+\frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} g^*(x')dx'-\frac{1}{2}\iint_D F^*(x', y')dx'dy'.$$

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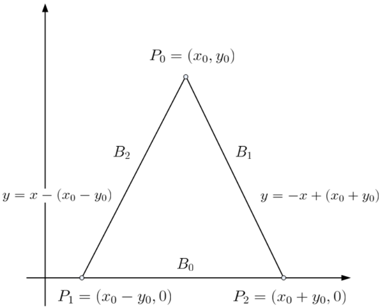

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Initial–boundary value problem for the wave equation

Oscillations of an elastic wire of length L are described by the wave equation

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad 0 < x < L, \quad t > 0, \tag{82}$$

with initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L. \tag{83}$$

- If the endpoints are fixed, then $u(x,t)$ satisfies Dirichletove boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t \ge 0. \tag{84}$$

- If the end points freely oscillate, then $u(x,t)$ satisfies Neumann conditions

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Theorem (Uniqueness of solution)

Let u_1 and u_2 be C^2 solutions of the wave equation (82) with initial conditions (83) and Dirichlet boundary conditions (84). Then $u_1 = u_2$.

Remark

The same proof applies to the wave equation with Neumann boundary conditions.

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Compatibility of initial and boundary conditions

$$f(0) = f(L) = 0, \quad g(0) = g(L) = 0. \tag{89}$$

By the method of separation of variables $u(x,t) = P(x)Q(t)$ we obtain a sequence of solutions

$$u_n(x,t) = P_n(x)Q_n(t) = \left[a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right] \sin\left(\frac{n\pi}{L} x\right) \tag{90}$$

called n–th order harmonics.

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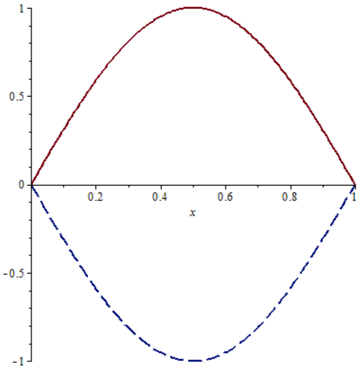
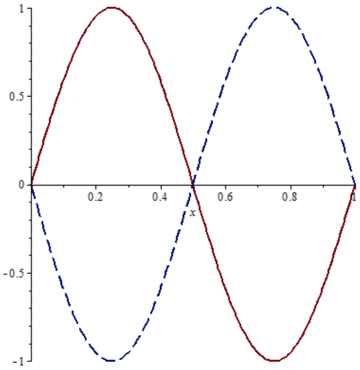
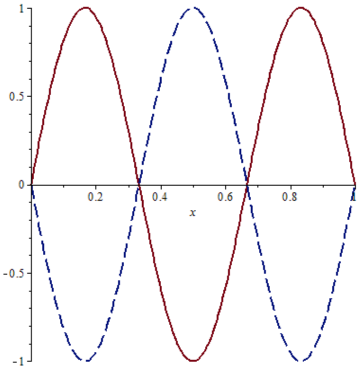


First harmonic



Second harmonic



Third harmonic

Theorem (Egzistence)

Let $f \in C^4([0,L])$ and $g \in C^3([0,L])$. Assume that f and g satisfy the following conditions

1. $f(0) = f(L) = 0$, $f''(0) = f''(L) = 0$,
2. $g(0) = g(L) = 0$.

Then

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right) \right] \sin\left(\frac{n\pi}{L} x\right), \tag{91}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx, \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx, \tag{92}$$

is a classical solution of the wave equation with Dirichlet boundary conditions

$$u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < L, \quad t > 0, \tag{93}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad 0 \le x \le L, \tag{94}$$

$$u(0,t) = u(L,t) = 0, \qquad t \ge 0. \tag{95}$$

Remark

If f and g od not satisfy the conditions in the theorem, the formal solution may not satisfy the wave equation.

Example

$$f(x) = \begin{cases} \frac{u_0}{x_0} x, & 0 \le x \le x_0, \\ u_0 \frac{x-L}{x_0-L}, & x_0 < x \le L, \end{cases} \tag{96}$$

$$g(x) = 0, \tag{97}$$

$$u(x,t) = \frac{2L^2}{\pi^2} \frac{u_0}{x_0(L-x_0)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x_0}{L}\right) \cos\left(\frac{n\pi c}{L} t\right) \sin\left(\frac{n\pi}{L} x\right) \tag{98}$$

The function u is well defined, but u_{xx} and u_{tt} diverge!

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Neumann boundary condtions

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \tag{99}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L, \tag{100}$$

$$u_x(0,t) = u_x(L,t) = 0, \quad t \ge 0. \tag{101}$$

Compatibility of initial and boundary conditions

$$f'(0) = f'(L) = 0, \quad g'(0) = g'(L) = 0. \tag{102}$$

Solution:

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right] \cos\left(\frac{n\pi}{L} x\right), \tag{103}$$

where a_n and b_n are determined from the boundary conditions:

$$a_0 = \frac{2}{L}\int_0^L f(x)dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \tag{104}$$

$$b_0 = \frac{2}{L}\int_0^L g(x)dx, \quad b_n = \frac{2}{n\pi c}\int_0^L g(x)\cos\left(\frac{n\pi}{L}x\right)dx. \tag{105}$$

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Neumann boundary conditions

$$u_{tt} - c^2 u_{xx} = F(x,t), \quad 0 < x < L, \quad t > 0, \tag{106}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 \le x \le L. \tag{107}$$

$F(x,t)$ models an external force acting on the wire.

$$u_x(0,t) = u_x(L,t) = 0, \quad t \ge 0. \tag{108}$$

We assume the solution in the form

$$u(x,t) = \frac{1}{2}Q_0(t) + \sum_{n=1}^{\infty} Q_n(t)\cos\left(\frac{n\pi}{L}x\right) \tag{109}$$

for some unknown functions $Q_n(t)$, $n \ge 0$. By substituting $u(x,t)$ into the wave equation we find

$$\frac{1}{2}Q_0''(t) + \sum_{n=1}^{\infty}\left[Q_n''(t) + \left(\frac{n\pi c}{L}\right)^2 Q_n(t)\right]\cos\left(\frac{n\pi}{L}x\right) = F(x,t). \tag{110}$$

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If $F(x,t)$ satisfies the conditions

$$F_x(0,t) = F_x(L,t) = 0, \quad t \geq 0, \tag{111}$$

then $F(x,t)$ can be written in the form

$$F(x,t) = \frac{1}{2}C_0(t) + \sum_{n=1}^{\infty} C_n(t)\cos\left(\frac{n\pi}{L}x\right), \quad 0 \leq x \leq L. \tag{112}$$

The wave equation yields

$$Q_0''(t) = C_0(t), \tag{113}$$

$$Q_n''(t) + \left(\frac{n\pi c}{L}\right)^2 Q_n(t) = C_n(t), \quad n \geq 1. \tag{114}$$

Solution:

$$Q_0(t) = a_0 + b_0 t + Q_0^p(t), \tag{115}$$

$$Q_n(t) = a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) + Q_n^p(t). \tag{116}$$

Solution of the nonhomogeneous equation:

$$u(x,t) = u_h(x,t) + u_p(x,t) \tag{117}$$

where

$$u_h(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right)\right] \cos\left(\frac{n\pi}{L} x\right), \tag{118}$$

$$u_p(x,t) = \frac{1}{2} Q_0^p(t) + \sum_{n=1}^{\infty} Q_n^p(t) \cos\left(\frac{n\pi}{L} x\right). \tag{119}$$

The Laplace equation

The Laplace equation is an elliptic equation of the form

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega \tag{120}$$

$\Omega \subseteq \mathbb{R}^2$ is a bounded domain (open, connected, bounded set)

$\partial\Omega$ is a union of simple, closed, piecewise smooth curves

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{Laplace operator} \tag{121}$$

Applications of Laplace equation

- Distrubution of electric potential in a domain free of charge.
- Stationary distribution of temperature in a heat conducting body.

Definition

A function $u \in C^2(\Omega)$ satisfying the Laplace equation in the domain $\Omega \subseteq \mathbb{R}^2$ is called a harmonic function in Ω.

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Boundary problem

A nonhomogeneous Laplace equation is called the Poisson equation:

$$u_{xx} + u_{yy} = f(x, y), \quad (x, y) \in \Omega. \tag{122}$$

Ω bounded domain

- Dirichlet boundary condition

$$u(x, y) = g(x, y), \quad (x, y) \in \partial\Omega \tag{123}$$

- Neumann boundary condtion

$$\frac{\partial u}{\partial \vec{n}}(x, y) = g(x, y), \quad (x, y) \in \partial\Omega \tag{124}$$

$$\frac{\partial u}{\partial \vec{n}} = \nabla \cdot \vec{n} \quad \text{directional derivative,} \tag{125}$$

$$\vec{n} \quad \text{unit normal vector at the point } (x, y) \in \partial\Omega \text{ pointing outwards.} \tag{126}$$

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In studying the Laplace equation we need the Gauss theorem.

Theorem (Gauss)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain whose boundary $\partial\Omega$ is a union of simple, closed piecewise shooth curves. Let $\vec{n}$ be a unit normal vector on the boundary $\partial\Omega$ pointing outwards. If $\vec{F}$ is a vector field of class C^1 in $\bar{\Omega}$, then

$$\int_{\partial\Omega} \vec{F} \cdot \vec{n} ds = \iint_{\Omega} (\nabla \cdot \vec{F})\, dx\, dy \tag{127}$$

where $\partial\Omega$ is positively oriented.

Lemma

Let Ω be a bounded domain in $\mathbb{R}^2$. If the Neumann problem

$$u_{xx} + u_{yy} = f(x,y), \quad (x,y) \in \Omega, \tag{128}$$

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has a solution, then f and g satisfy the consistency condition

$$\int_{\partial\Omega} g\, ds = \iint_{\Omega} f\, dxdy. \tag{130}$$

- Note that for the Laplace equation we have

$$f = 0 \quad \Rightarrow \quad \int_{\partial\Omega} g\, ds = 0. \tag{131}$$

- Thus, a harmonic function u satisfies the condition

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Theorem (Weak maximum principle)

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic function in a bounded domain $\Omega \subset \mathbb{R}^2$, then

$$\max_{\bar{\Omega}} u = u(x', y') \tag{133}$$

at some point $(x', y') \in \partial\Omega$. In other words, the function u attains its maximum in $\bar{\Omega}$ at a boundary point $(x', y') \in \partial\Omega$.

Theorem (Weak maximum principle

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic function in a bounded domain $\Omega \subseteq \mathbb{R}^2$, then

$$\min_{\bar{\Omega}} u = u(x', y') \tag{134}$$

at some point $(x', y') \in \partial\Omega$.

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Corollary

If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic function in a bounded domain $\Omega \subseteq \mathbb{R}^2$ and $u(x, y) = 0$ for all $(x, y) \in \partial\Omega$, then $u = 0$.

Theorem (Uniqueness of solution for Dirichlet problem)

Assume Ω is a bounded domain in $\mathbb{R}^2$. Then there exists at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of the Dirichlet problem

$$\Delta u(x, y) = f(x, y), \quad (x, y) \in \Omega, \tag{136}$$

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Stability of solutions of the Dirichlet problem

Let Ω be a bounded domain in $\mathbb{R}^2$. Assume $u_1, u_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ are two solutions of the Poisson equation

$$\Delta u_1(x,y) = f(x,y), \quad \Delta u_2(x,y) = f(x,y), \quad (x,y) \in \Omega, \tag{138}$$

satisfying the boundary conditions

$$u_1(x,y) = g_1(x,y) \quad u_2(x,y) = g_2(x,y), \quad (x,y) \in \partial\Omega, \tag{139}$$

where $g_1, g_2 \in C(\partial\Omega)$. If

$$\max_{\partial\Omega} |g_1 - g_2| < \varepsilon, \quad \text{then} \quad \max_{\bar{\Omega}} |u_1 - u_2| < \varepsilon. \tag{140}$$

Theorem (Mean value principle)

Let u be a harmonic function in the domain Ω (which may not be bounded), and let $\bar{K}_r(x_0, y_0) \subset \Omega$ be a closed disk of radius $r > 0$ centered at $(x_0, y_0) \in \Omega$. Then

$$u(x_0, y_0) = \frac{1}{2\pi r} \int_{C_r} u \, ds \tag{141}$$

where C_r is a circle of radius $r > 0$ centered at (x_0, y_0).

Theorem

Assume the function $u \in C^2(\Omega)$ satisfies the Mean value principle at every point in the domain Ω. Then u is a harmonic function in Ω.

Theorem (Strong maximum principle)

Assume u is a harmonic function in the domain Ω (which may not be bounded). If u attains a minimum or maximum value at an interior point of Ω, then u is a constant function.

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The method of separation of variables for the Laplace equation yileds solution in the form of a series $u = \sum_{n=1}^{\infty} u_n$ where u_n are harmonic functions in a domain Ω.

The exact form of the solution depends on

- the geometry of Ω,
- boundary conditions on $\partial\Omega$.

Problem: Under which conditions does the formal solution $u = \sum_{n=1}^{\infty} u_n$ represent a harmonic function in Ω?

Theorem

Assume Ω is a bounded domain in $\mathbb{R}^2$. Let $u = \sum_{n=1}^{\infty} u_n$ be a formal solution of the Dirichlet problem

$$\Delta u(x, y) = 0, \quad (x, y) \in \Omega, \tag{142}$$

$$u(x, y) = g(x, y), \quad (x, y) \in \partial\Omega, \tag{143}$$

where $g \in C(\partial\Omega)$ and $u_n \in C^2(\Omega) \cap C(\bar{\Omega})$ are harmonic functions in Ω for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} u_n$ converges uniformly to g on $\partial\Omega$, the $\sum_{n=1}^{\infty} u_n$ converges uniformly in $\bar{\Omega}$, i $u = \sum_{n=1}^{\infty} u_n$ is a classical solution of (142)–(143).

Separation of variables for Laplace equation

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$$\Delta u(x,y) = 0, \quad (x,y) \in \Omega, \tag{142}$$

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega, \tag{143}$$

where $g \in C(\partial\Omega)$ and $u_n \in C^2(\Omega) \cap C(\bar{\Omega})$ are harmonic functions in Ω for all $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} u_n$ converges uniformly to g on $\partial\Omega$, the $\sum_{n=1}^{\infty} u_n$ converges uniformly in $\bar{\Omega}$, i $u = \sum_{n=1}^{\infty} u_n$ is a classical solution of (142)–(143).

The method of separation of variables for the Laplace equation yileds solution in the form of a series $u = \sum_{n=1}^{\infty} u_n$ where u_n are harmonic functions in a domain Ω.

The exact form of the solution depends on

- the geometry of Ω,
- boundary conditions on $\partial\Omega$.

Problem: Under which conditions does the formal solution $u = \sum_{n=1}^{\infty} u_n$ represent a harmonic function in Ω?

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Separation of variables for Laplace equation

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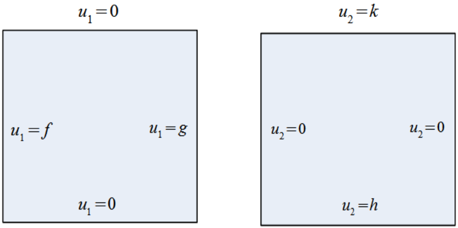
Dirichlet problem in a rectangular domain

$$\Delta u(x,y) = 0, \quad 0 < x < b, \quad 0 < y < d, \tag{144}$$

$$u(x,0) = h(x), \quad u(x,d) = k(x), \quad 0 \le x \le b, \tag{145}$$

$$u(0,y) = f(y), \quad u(b,y) = g(y), \quad 0 \le y \le d. \tag{146}$$

- We want to write solution of the Laplace equation as a series using the eingenfunctions of the associated Sturm–Liouville problem.
- We look for the solution in the form $u = u_1 + u_2$ where u_1 and u_2 satisfy homogeneous boundary conditions on the opposite sides the rectagnle.



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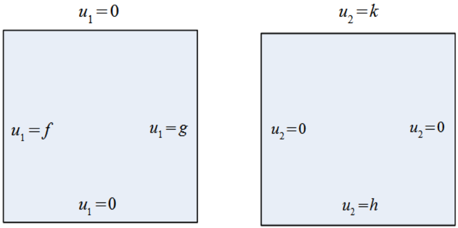

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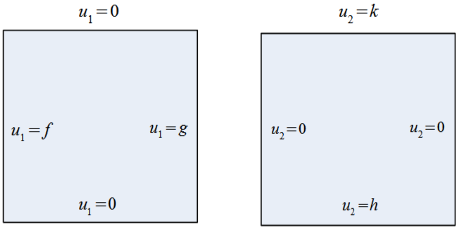

By separating the variatibles in the equation for u_1 we obtain

$$u_1(x,y)=\sum_{n=1}^{\infty}\left[A_n \mathsf{sh}\left(\frac{n\pi}{d}x\right)+B_n \mathsf{sh}\left(\frac{n\pi}{d}(x-b)\right)\right]\sin\left(\frac{n\pi}{d}y\right), \tag{147}$$

$$A_n=\frac{2}{d\,\mathsf{sh}\left(\frac{n\pi b}{d}\right)}\int_0^d g(y)\sin\left(\frac{n\pi}{d}y\right)dy, \tag{148}$$

$$B_n=-\frac{2}{d\,\mathsf{sh}\left(\frac{n\pi b}{d}\right)}\int_0^d f(y)\sin\left(\frac{n\pi}{d}y)\right)dy. \tag{149}$$

Similarly, we find

$$u_2(x,y) = \sum_{n=1}^{\infty}\left[C_n \mathsf{sh}\left(\frac{n\pi}{b}y\right) + D_n \mathsf{sh}\left(\frac{n\pi}{b}(y-d)\right)\right]\sin\left(\frac{n\pi}{b}x\right), \tag{150}$$

$$C_n = \frac{2}{b\,\mathsf{sh}\left(\frac{n\pi d}{b}\right)}\int_0^b k(x)\sin\left(\frac{n\pi}{b}x\right)dx, \tag{151}$$

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Harmonic polynomials

If the functions f, g, h and k do not vanish at the vertices of the rectangle $\Omega = (0,b) \times (0,d)$, then the Dirichlet problem can be solved by adding harmonic polynomials to the boundary conditions.

Initial problem

$$\nabla^2 u(x,y) = 0, \quad (x,y) \in \Omega, \tag{153}$$

$$u(x,y) = G(x,y), \quad (x,y) \in \partial\Omega, \quad G \in C(\partial\Omega). \tag{154}$$

Modified problem

$$\nabla^2 \Big(u(x,y) - P_2(x,y) \Big) = 0, \quad (x,y) \in \Omega, \tag{155}$$

$$u(x,y) - P_2(x,y) = G(x,y) - P_2(x,y), \quad (x,y) \in \partial\Omega \tag{156}$$

where

$$P_2(x,y) = a_1(x^2 - y^2) + a_2 xy + a_3 x + a_4 y + a_5, \quad a_i \in \mathbb{R}. \tag{157}$$

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Note that $P_2(x, y)$ is a *harmonic polynomial* because

$$\nabla^2 P_2(x, y) = 0 \quad \text{for any choice of the coefficients } a_i \in \mathbb{R}. \tag{158}$$

Define

$$v(x, y) = u(x, y) - P_2(x, y), \quad \tilde{G}(x, y) = G(x, y) - P_2(x, y). \tag{159}$$

Dirichlet problem for v:

$$\nabla^2 v(x, y) = 0, \quad (x, y) \in \Omega, \tag{160}$$

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The polynomial P_2 can be chosen such that $\tilde{G}(x, y) = G(x, y) - P_2(x, y)$ vanishes at the vertices of the rectangle Ω:

$$G(0, 0) - P_2(0, 0) = 0, \tag{162}$$

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The boundary conditions for v are continuous on $\partial\Omega$, hence the Laplace equation for v can be solved by the method of separation of variables.

Solution of the initial problem: $u(x, y) = v(x, y) + P_2(x, y)$.

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