

Partial Differential Equations
Class notes

Saša Krešić–Jurić
Faculty of Science
University of Split

2006.

Contents

1	Introduction	3
1.1	Basic notions	3
1.2	Linear equations and the superposition principle	5
1.3	Classical equations of mathematical physics	7
1.4	Elementary techniques	9
1.5	Initial and boundary conditions	13
1.6	Stability of solutions	14
1.7	Problems	18
2	Fourier series	20
2.1	Expansion of a function into a Fourier series	20
2.2	Convergence of the Fourier series	26
2.2.1	Uniform convergence	31
3	Quasi-linear first-order equations	42
4	Classification of second-order equations	46
4.1	Canonical form of hyperbolic equations	50
4.2	Canonical form of parabolic equations	53
4.3	Canonical form of elliptic equations	56
5	The Heat Equation	60
5.1	Maximum Principle and Uniqueness of Solutions	60
5.2	Separation of Variables for the Homogeneous Equation	66

<i>CONTENTS</i>	2
5.3 Separation of Variables for the Nonhomogeneous Equation	76
6 The Wave Equation	81
6.1 Wave Motion and d'Alembert's Solution	81
6.2 D'Alembert's Solution for the Nonhomogeneous Wave Equation . . .	86
6.3 Initial–Boundary Value Problem for the Wave Equation	90
6.3.1 Separation of Variables for the Homogeneous Equation	92
6.3.2 Separation of Variables for the Nonhomogeneous Equation . .	102
7 Laplace's Equation	106
7.1 General Properties of Laplace's Equation	107
7.2 Separation of Variables for Laplace's Equation	115
7.2.1 Rectangular Domains	116
7.2.2 Circular Domains	123
7.2.3 Poisson's Formula	126

Chapter 1

Introduction

1.1 Basic notions

Partial differential equations (PDEs) describe relations between an unknown function u and its partial derivatives. These equations are very important in physics and engineering because they model various phenomena occurring in nature. In recent years, partial differential equations have also found applications in biology, chemistry, computer science, and economics.

Let $u(x) = u(x_1, x_2, \dots, x_n)$ be a function of n independent variables x_1, x_2, \dots, x_n . We denote the partial derivatives by

$$u_{x_i} = \frac{\partial u}{\partial x_i}, \quad u_{x_i x_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad \dots \quad (1.1)$$

Definition 1.1 We say that the function u is of class C^k on Ω , and we write $u \in C^k(\Omega)$, if u has continuous partial derivatives of order k on Ω .

If $u \in C^k(\Omega)$, then the order in which the partial derivatives are taken is not important. Note that $u \in C^k(\Omega)$ implies $u \in C^{k-1}(\Omega)$. The set of continuous functions on Ω is denoted by $C^0(\Omega)$.

Definition 1.2 A partial differential equation is an equation of the form

$$F(x_1, \dots, x_n, u, u_{x_1}, \dots, u_{x_n}, u_{x_1 x_1}, u_{x_1 x_2}, \dots) = 0 \quad (1.2)$$

where $u = u(x_1, x_2, \dots, x_n)$ is an unknown function of the independent variables x_1, x_2, \dots, x_n .

Partial differential equations are usually considered on an open connected set $\Omega \subseteq \mathbb{R}^n$. The order of the equation is the order of the highest partial derivative that appears in the equation.

Definition 1.3 *A solution of the partial differential equation (1.2) of order $k > 0$ on the set $\Omega \subseteq \mathbb{R}^n$ is a function $u \in C^k(\Omega)$ that satisfies equation (1.2) at every point of the set Ω .*

These solutions are called *classical* or *strong* solutions. In applications, so-called distributional and weak solutions are also of interest, but we will not consider them here.

Example 1.1 *The equation*

$$u_{xx} - u_{yy} = 0 \tag{1.3}$$

is a second-order partial differential equation. It is easy to verify that the functions $u(x, y) = (x + y)^3$ and $u(x, y) = \sin(x - y)$ are solutions of equation (1.3) on the set $\Omega = \mathbb{R}^2$.

Example 1.2 *The Korteweg–de Vries equation*

$$u_t + u_{xxx} - 6uu_x = 0 \tag{1.4}$$

models waves on the surface of water in a shallow channel. Verify that the function

$$u(x, t) = \frac{c}{2} \operatorname{sch}^2 \left[\frac{\sqrt{2}}{2} (x - ct - x_0) \right], \quad c > 0, x_0 \in \mathbb{R} \tag{1.5}$$

satisfies equation (1.5), where $\operatorname{sch}(x) = 2/(e^x + e^{-x})$ is the hyperbolic secant, on the set $\Omega = \mathbb{R}^2$. This function describes a soliton wave that travels without dispersion with speed $c > 0$.

Partial differential equations can be roughly classified according to the following properties.

1. Order of the equation

The basic classification of partial differential equations is according to the order of the equation. In general, the higher the order of the equation, the more difficult it is to solve.

2. Linear and nonlinear equations

- (a) Equation (1.2) is *linear* if F is a linear function of the variables u and all of its partial derivatives. In this case the coefficients multiplying u and its derivatives depend only on the independent variables x_1, \dots, x_n .
- (b) The equation is *nonlinear* if it is not linear.
- (c) Equation (1.2) is *quasi-linear* if F is linear in all highest-order partial derivatives of u .

For example, Euler's equation

$$xu_x + yu_y = nu, \quad n \in \mathbb{N}, \quad (1.6)$$

is a linear first-order equation because it is linear in the variables u_x and u_y . The equation

$$u_x u_{xx} + xu u_y = \sin(y) \quad (1.7)$$

is a quasi-linear second-order equation because it is linear in the highest derivative u_{xx} . The equation

$$u_x u_{xx}^2 + xu u_y = \sin(y) \quad (1.8)$$

is nonlinear because the term u_{xx}^2 is not linear. Let us list a few more examples:

$$u u_{xy} + u_x = y, \quad \text{quasi-linear second-order equation,} \quad (1.9)$$

$$(u_x)^2 + (u_y)^2 = 1, \quad \text{nonlinear first-order equation,} \quad (1.10)$$

$$u_x u_{xxy} + x u_y = \sin(y), \quad \text{quasi-linear third-order equation,} \quad (1.11)$$

$$u_t + u_{xxx} - 6u u_x = 0, \quad \text{quasi-linear third-order equation.} \quad (1.12)$$

1.2 Linear equations and the superposition principle

We will be particularly interested in linear second-order partial differential equations, since this type of equation has important applications in the natural and technical sciences. Classical equations of mathematical physics such as the wave equation, the

heat equation, Poisson's equation, and Schrödinger's equation are linear second-order equations. A linear second-order equation in n independent variables x_1, x_2, \dots, x_n has the general form

$$\sum_{i,j=1}^n A_{ij}u_{x_i x_j} + \sum_{i=1}^n B_i u_{x_i} + Fu = G \quad (1.13)$$

where A_{ij} , B_i , F , and G are functions of the variables x_1, x_2, \dots, x_n . If u is a classical solution of equation (1.13), then $u_{x_i x_j} = u_{x_j x_i}$, so equation (1.13) can be reduced to a form in which $A_{ij} = A_{ji}$. To the differential equation (1.13) we can associate the differential operator

$$L = \sum_{i,j=1}^n A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n B_i \frac{\partial}{\partial x_i} + F \quad (1.14)$$

where F denotes the operator of multiplication by the function F . Then the equation can be written in the compact form $L[u] = G$.

Definition 1.4 *The equation $L[u] = G$ is called homogeneous if $G = 0$. Otherwise, we say that the equation is nonhomogeneous.*

The operator L is linear because

$$L[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 L[u_1] + \alpha_2 L[u_2], \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}. \quad (1.15)$$

Linear equations have the important property called the *superposition principle*. If u_1 and u_2 are solutions of the equations

$$L[u_1] = G_1, \quad L[u_2] = G_2, \quad (1.16)$$

then the linear combination $u = \alpha_1 u_1 + \alpha_2 u_2$, $\alpha_i \in \mathbb{R}$, is a solution of the equation

$$L[u] = \alpha_1 L[u_1] + \alpha_2 L[u_2] = \alpha_1 G_1 + \alpha_2 G_2. \quad (1.17)$$

In the special case when u_1 and u_2 are solutions of the homogeneous equation $L[u] = 0$, every linear combination $u = \alpha_1 u_1 + \alpha_2 u_2$ is also a solution of the same equation, since $L[\alpha_1 u_1 + \alpha_2 u_2] = 0$. This principle is particularly important in solving partial differential equations by the method of separation of variables, since the general solution can be written as a linear combination of certain particular solutions. We will consider this method in more detail in the following chapters.

Example 1.3 *Let us consider the equation*

$$u_{tt} - u_{xx} = 0. \quad (1.18)$$

It is easy to verify that the functions $u_n(x, t) = \sin(nt) \cos(nx)$ are solutions of the equation for every $n \in \mathbb{N}$. Therefore, every linear combination

$$u(x, t) = \sum_{n=1}^N c_n \sin(nt) \cos(nx), \quad c_n \in \mathbb{R}, \quad (1.19)$$

is also a solution of equation (1.18).

Example 1.4 *Determine a solution of the equation*

$$u_{xx} - u_y = 18x + 8y \quad (1.20)$$

using the superposition principle.

Solution. Let us consider the two equations

$$u_{xx} - u_y = 18x, \quad (1.21)$$

$$u_{xx} - u_y = 8y. \quad (1.22)$$

We look for solutions of these equations in the form $u_1 = u_1(x)$ and $u_2 = u_2(y)$, respectively. Then $u_1''(x) = 18x$ and $-u_2'(y) = 8y$, so by integration we obtain

$$u_1(x) = 3x^3 + ax + b, \quad u_2(y) = -4y^2 + c \quad (1.23)$$

where $a, b, c \in \mathbb{R}$ are constants of integration. By the superposition principle, the function

$$u(x, y) = u_1(x) + u_2(y) = 3x^3 - 4y^2 + ax + d, \quad (1.24)$$

where $d = b + c$, is a solution of equation (1.20). \square

1.3 Classical equations of mathematical physics

Fundamental laws in the natural sciences are often formulated in the form of partial differential equations. When a physical quantity $u = u(x, y, z, t)$ depends on spatial

or temporal changes of u , then the function u satisfies a partial differential equation. In most cases these are linear second-order equations that depend on the spatial variables x, y, z and the time variable t . Let us list some important equations of mathematical physics.

- (1) **Wave equation.** Wave motion in different media is described by the wave equation

$$u_{tt} - c^2 \nabla^2 u = 0 \quad (1.25)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (1.26)$$

is the *Laplace operator*. For example, if $u(x, y, z, t)$ denotes the air pressure at the point (x, y, z) at time t and c is the speed of sound, then equation (1.25) describes the propagation of sound waves. The same equation describes the propagation of electromagnetic waves when u represents a scalar component of the electric or magnetic field, and c is the speed of light.

- (2) **Heat equation.** Let $u(x, y, z, t)$ be the temperature of a homogeneous heat-conducting body that has no internal heat sources. Then the function u satisfies the heat equation

$$u_t - k \nabla^2 u = 0 \quad (1.27)$$

where the constant $k > 0$ is the thermal conductivity of the material. This equation is also called the *diffusion equation* because it describes diffusion processes in matter.

- (3) **Laplace's equation.** If the temperature in a heat-conducting body is stationary, then $u_t = 0$, so the heat equation reduces to Laplace's equation

$$\nabla^2 u = 0. \quad (1.28)$$

This equation also describes the distribution of the electric potential u in a region of space without charges.

The above equations represent the so-called canonical or standard forms of second-order partial differential equations, which we will study in detail in the following

chapters. Let us list a few more second-order equations that have important applications in physics.

(4) **Poisson's equation**

$$\nabla^2 u = \rho(x, y, z), \quad (1.29)$$

(5) **Helmholtz equation**

$$\nabla^2 u + \lambda u = 0, \quad (1.30)$$

(6) **Schrödinger equation**

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(x, y, z)\psi = i \hbar \psi_t. \quad (1.31)$$

1.4 Elementary techniques

In simple cases, partial differential equations can be solved by direct integration or by introducing new variables. If the equation describes the behavior of a physical system possessing a certain symmetry, for example if the system is invariant under rotations or some other transformation, then the equation can often be simplified by exploiting these symmetries. Recall that the solutions of ordinary differential equations depend on arbitrary constants of integration. Similarly, the general solution of a partial differential equation depends on arbitrary functions that arise in the process of integration. The following examples illustrate these properties of partial differential equations.

Example 1.5 *Determine the solution of the equation*

$$u_{yy} = 2 \quad (1.32)$$

for the function $u = u(x, y)$.

Solution. Integrating with respect to the variable y we obtain

$$u_y = \int 2 \, dy = 2y + f(x). \quad (1.33)$$

Repeating the procedure we find

$$u(x, y) = \int (2y + f(x)) dy = y^2 + yf(x) + g(x) \quad (1.34)$$

where $f(x)$ and $g(x)$ are arbitrary functions. Since the function u is a classical solution of equation (1.32), f and g are functions of class C^2 . \square

Example 1.6 Determine the general solution of the PDE

$$u_x - u_y = 0. \quad (1.35)$$

Solution. Introduce new variables $\alpha = x + y$ and $\beta = x - y$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}, \quad (1.36)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \alpha} \frac{\partial \alpha}{\partial y} + \frac{\partial u}{\partial \beta} \frac{\partial \beta}{\partial y} = \frac{\partial u}{\partial \alpha} - \frac{\partial u}{\partial \beta}. \quad (1.37)$$

Subtracting we obtain

$$u_x - u_y = 2u_\beta = 0. \quad (1.38)$$

In the new variables the equation has the simpler form

$$u_\beta = 0. \quad (1.39)$$

By integrating the equation we obtain

$$u = f(\alpha) = f(x + y) \quad (1.40)$$

where $f(\alpha)$ is an arbitrary C^1 function. For example, each of the functions $e^{n(x+y)}$, $\sin(n(x+y))$ and $\cos(n(x+y))$, $n \in \mathbb{N}$, is a solution of the equation. \square

Example 1.7 Determine the general solution of the problem

$$u_{xx} + cu = 0, \quad (1.41)$$

where $u = u(x, t)$, for $c > 0$, $c = 0$ and $c < 0$.

Solution. Since the function $u(x, t)$ depends on the variables x and t , equation (1.41) can be regarded as an ordinary differential equation in the variable x depending on the parameter t . If $c > 0$, the general solution is given by

$$u(x, t) = C_1(t) \sin(\sqrt{c}x) + C_2(t) \cos(\sqrt{c}x) \quad (1.42)$$

where $C_1(t)$ and $C_2(t)$ are arbitrary functions of class C^2 . Similarly, for $c = 0$ we have

$$u(x, t) = C_1(t)x + C_2(t), \quad (1.43)$$

while for $c < 0$ the solution is given by

$$u(x, t) = C_1(t)e^{\sqrt{|c|x}} + C_2(t)e^{-\sqrt{|c|x}}. \quad (1.44)$$

□

Example 1.8 Determine the general solution of the equation

$$u_x = u^2 \quad (1.45)$$

for the function $u = u(x, y)$.

Solution. As in the previous example, equation (1.45) can be regarded as an ordinary differential equation in the variable x with parameter y . By separating the equation we obtain $u^{-2}du = dx$, from which it follows that $-u^{-1} = x + f(y)$. Hence,

$$u(x, y) = -\frac{1}{x + f(y)} \quad (1.46)$$

where $f(y)$ is an arbitrary function of class C^1 . □

The following example illustrates how the symmetry of the problem can be used to reduce a partial differential equation to a simpler form.

Example 1.9 Determine a spherically symmetric solution of Laplace's equation

$$u_{xx} + u_{yy} + u_{zz} = 0. \quad (1.47)$$

Solution. The desired solution is invariant under the rotation group $SO(3)$ and therefore depends only on the distance of the point (x, y, z) from the origin. Hence we seek $u(x, y, z)$ in the form

$$u = f(r), \quad r = \sqrt{x^2 + y^2 + z^2}. \quad (1.48)$$

Now

$$u_x = f'(r) \frac{\partial r}{\partial x} = f'(r) \frac{x}{r}, \quad (1.49)$$

$$u_{xx} = \frac{\partial}{\partial x} \left(f'(r) \frac{x}{r} \right) + f'(r) \frac{\partial}{\partial x} \left(\frac{x}{r} \right) \quad (1.50)$$

$$= f''(r) \left(\frac{x}{r} \right)^2 + f'(r) \left(\frac{1}{r} - \frac{x^2}{r^3} \right). \quad (1.51)$$

By symmetry of the function u , the remaining derivatives are given by analogous expressions,

$$u_{yy} = f''(r) \left(\frac{y}{r} \right)^2 + f'(r) \left(\frac{1}{r} - \frac{y^2}{r^3} \right), \quad (1.52)$$

$$u_{zz} = f''(r) \left(\frac{z}{r} \right)^2 + f'(r) \left(\frac{1}{r} - \frac{z^2}{r^3} \right). \quad (1.53)$$

Adding equations (1.51)–(1.53), we obtain

$$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r) = 0. \quad (1.54)$$

Introduce the substitution $g(r) = f'(r)$. Then $g'(r) + 2r^{-1}g(r) = 0$, which implies $g(r) = Cr^{-2}$. Now

$$f(r) = \int g(r) dr = -\frac{C}{r} + K \quad (1.55)$$

where $C, K \in \mathbb{R}$ are arbitrary constants of integration. Therefore, a spherically symmetric solution of Laplace's equation is given by

$$u(x, y, z) = -\frac{C}{\sqrt{x^2 + y^2 + z^2}} + K. \quad (1.56)$$

□

1.5 Initial and boundary conditions

From the previous discussion it is clear that partial differential equations may have infinitely many solutions that depend on arbitrary functions. If an equation models a physical phenomenon, then it is necessary to have a unique solution in order to predict the behavior of the system. Therefore partial differential equations are usually considered together with prescribed boundary and/or initial conditions that make the solution unique. Using concrete examples we will illustrate partial differential equations together with the corresponding boundary and initial conditions and explain their physical meaning.

Let us consider the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \quad t > 0. \quad (1.57)$$

where the function $u(x, t)$ represents the amplitude of oscillation at the point x at time t . This equation holds under the assumption that there are no dissipative effects and that each point moves perpendicularly to the x -axis. The oscillation of the string is a dynamical problem that requires knowledge of the initial position and the initial velocity of the string at every point $x \in [0, L]$. Therefore, we require that the function u satisfy the initial conditions

$$u(x, 0) = f(x), \quad x \in [0, L], \quad (1.58)$$

$$u_t(x, 0) = g(x), \quad x \in [0, L], \quad (1.59)$$

where $f(x)$ is the initial displacement and $g(x)$ is the initial velocity at the point x . The solution of the wave equation also depends on the boundary conditions at the ends of the string. If the string is fixed at the points $x = 0$ and $x = L$, then u satisfies the *Dirichlet* boundary conditions

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \quad (1.60)$$

If the ends of the string oscillate freely in the direction perpendicular to the x -axis, then the derivative of the amplitude vanishes at $x = 0$ and $x = L$, so in this case the function u satisfies the *Neumann* boundary conditions

$$u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0. \quad (1.61)$$

Furthermore, if the string is bent into a circle so that the point $x = 0$ coincides with the point $x = L$, then the function u satisfies the *periodic* conditions

$$u(0, t) = u(L, t), \quad (1.62)$$

$$u_x(0, t) = u_x(L, t), \quad t \geq 0. \quad (1.63)$$

Boundary conditions can also be combined; for example, u may satisfy a Dirichlet condition at one end and a Neumann condition at the other end of the interval.

Let us now consider Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega, \quad (1.64)$$

on a domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is a simple, closed, piecewise smooth curve. If the value of the function u is known on the boundary $\partial\Omega$, then u satisfies the Dirichlet boundary condition

$$u(x, y) = h(x, y), \quad (x, y) \in \partial\Omega. \quad (1.65)$$

If the normal derivative on the curve $\partial\Omega$ is prescribed, then u satisfies the Neumann boundary condition

$$\frac{\partial u}{\partial \vec{n}}(x, y) = h(x, y), \quad (x, y) \in \partial\Omega, \quad (1.66)$$

where \vec{n} is the unit outward normal vector to $\partial\Omega$, and

$$\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n} \quad (1.67)$$

is the directional derivative in the direction of the vector \vec{n} . In the following chapters we will see that for each of the above boundary and/or initial conditions there exists a unique solution of the corresponding partial differential equation.

1.6 Stability of solutions

In real problems, the initial and boundary conditions are usually determined by measurements, so these quantities are not known with complete accuracy. An error in the initial or boundary conditions may therefore produce an error in the solution of

the equation that need not be small. Furthermore, partial differential equations on complicated domains are often solved by numerical methods, so the discretization of the equation affects the accuracy of the solution. One of the fundamental theoretical questions is whether the mathematical problem, consisting of a partial differential equation together with initial or boundary conditions, is well posed. This question was formulated by the French mathematician Jacques Hadamard (1865–1963). According to his definition, a problem is well posed if it satisfies the following conditions:

- (1) *existence*: the problem has a solution,
- (2) *uniqueness*: the solution is unique for the given initial and/or boundary conditions,
- (3) *stability*: the solution depends continuously on the parameters of the equation and on the boundary or initial conditions.

We say that a partial differential equation is stable if small perturbations of the initial or boundary conditions produce small changes in the solution. In that case, the obtained solution will be a good approximation of the exact solution. The classical equations of mathematical physics described in Section 1.3 are well-posed problems, whereas in engineering we often encounter equations that are not stable. Let us illustrate these ideas with the following examples.

Laplace's equation

Let us explain how stability is formulated for Laplace's equation on a domain $\Omega \subset \mathbb{R}^2$,

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega, \quad (1.68)$$

with the Dirichlet boundary condition

$$u(x, y) = f(x, y), \quad (x, y) \in \partial\Omega. \quad (1.69)$$

Let u_1 and u_2 be solutions of Laplace's equation that satisfy the boundary conditions $u_1(x, y) = f_1(x, y)$ and $u_2(x, y) = f_2(x, y)$ for $(x, y) \in \partial\Omega$. We say that the solution

depends continuously on the boundary condition (1.69) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{(x,y) \in \partial\Omega} |f_1(x,y) - f_2(x,y)| < \delta \quad \Rightarrow \quad \sup_{(x,y) \in \bar{\Omega}} |u_1(x,y) - u_2(x,y)| < \varepsilon, \quad (1.70)$$

where $\bar{\Omega} = \Omega \cup \partial\Omega$ denotes the closure of the set Ω . Solutions that satisfy this condition are called *stable* solutions. In other words, the solution u is stable if a small change in the boundary condition on the curve $\partial\Omega$ produces a small change in the solution on the set Ω .

Hadamard's example

The following example illustrates that for improperly posed boundary conditions the solution of Laplace's equation need not be stable. In that case we say that the problem is not well posed. Consider the equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \mathbb{R}^2 \quad (1.71)$$

with boundary conditions on the line $y = 0$:

$$u(x, 0) = f(x), \quad u_y(x, 0) = g(x). \quad (1.72)$$

Let $v(x, y)$ be a solution of the same equation with modified boundary conditions

$$v(x, 0) = f(x), \quad v_y(x, 0) = g(x) + \frac{1}{n} \sin(nx). \quad (1.73)$$

It is clear that for sufficiently large $n > 0$ the difference in the boundary conditions can be made arbitrarily small since

$$\sup_{x \in \mathbb{R}} \left(|v(x, 0) - u(x, 0)| + |v_y(x, 0) - u_y(x, 0)| \right) = \frac{1}{n} \sup_{x \in \mathbb{R}} |\sin(nx)| \leq \frac{1}{n}. \quad (1.74)$$

However, the difference in the corresponding solutions on the domain \mathbb{R}^2 is large regardless of the value of the parameter n . Define the function $w = v - u$. Then w satisfies Laplace's equation

$$w_{xx} + w_{yy} = 0 \quad (1.75)$$

with boundary conditions

$$w(x, 0) = 0, \quad w_y(x, 0) = \frac{1}{n} \sin(nx). \quad (1.76)$$

It is easy to verify that the solution of the problem (1.75)–(1.76) is given by

$$w(x, y) = \frac{1}{n^2} \operatorname{sh}(ny) \sin(nx). \quad (1.77)$$

The function $\operatorname{sh}(ny)$ is unbounded on \mathbb{R} since $|\operatorname{sh}(ny)| \rightarrow \infty$ as $y \rightarrow \pm\infty$. Therefore, for every $x \in \mathbb{R}$ such that $\sin(nx) \neq 0$ we have

$$|w(x, y)| \rightarrow \infty \quad \text{as } y \rightarrow \pm\infty. \quad (1.78)$$

This shows that the function $w = v - u$ is unbounded on \mathbb{R}^2 . The graph of the function w is shown in Figure (1.1). We conclude that, although the difference in the initial conditions can be made arbitrarily small for sufficiently large $n > 0$, the corresponding solutions differ significantly when $|y|$ is sufficiently large. Therefore the problem (1.71)–(1.72) is not well posed.

Backward heat equation

Let us now consider the equation

$$u_t + u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (1.79)$$

$$u(x, 0) = 1. \quad (1.80)$$

This equation is obtained from the heat equation $u_t - u_{xx} = 0$ by replacing the variable t with $-t$. Let $v(x, t)$ be a solution of equation (1.79) with the initial condition

$$v(x, 0) = 1 + \frac{1}{n} \sin(nx). \quad (1.81)$$

It is easy to verify that the solutions $u(x, t)$ and $v(x, t)$ are given by

$$u(x, t) = 1, \quad v(x, t) = 1 + \frac{1}{n} e^{n^2 t} \sin(nx). \quad (1.82)$$

The difference in the initial conditions can be made arbitrarily small since

$$\sup_{x \in \mathbb{R}} |u(x, 0) - v(x, 0)| = \frac{1}{n} \sup_{x \in \mathbb{R}} |\sin(nx)| \leq \frac{1}{n}. \quad (1.83)$$

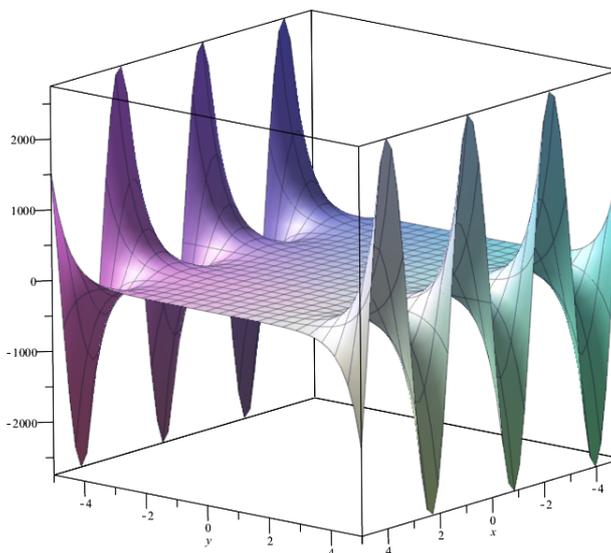


Figure 1.1: The graph of the function $w(x, y) = \frac{1}{n^2} \text{sh}(ny) \sin(nx)$ for $n = 2$.

However, the difference of the solutions $w = v - u$ is not bounded on \mathbb{R}^2 since

$$|w(x, t)| = \frac{1}{n} e^{n^2 t} |\sin(nx)| \rightarrow \infty \quad \text{as } t \rightarrow \infty \quad (1.84)$$

for every x such that $\sin(nx) \neq 0$. Hence, the backward heat equation together with the boundary condition (1.80) is not a well-posed problem.

1.7 Problems

1. Show that the equation $u_{xy} + u_x = 0$ has the general solution

$$u(x, y) = D(x)e^{-y} + E(y). \quad (1.85)$$

Hint: use the substitution $v = u_x$.

2. Determine the constants a and b so that the function $u(x, y) = f(ax + by)$ is a solution of the equation

$$u_x + 3u_y = 0. \quad (1.86)$$

3. Solve the equation

$$u_{xx} + u_{yy} = 5e^{x-2y}. \quad (1.87)$$

Hint: assume a solution of the form $u(x, y) = Ce^{ax+by}$.

4. Solve the system of equations

$$u_x = 3x^2y + y, \quad (1.88)$$

$$u_y = x^3 + x. \quad (1.89)$$

5. Show that the equation

$$u_t + u_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (1.90)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0, \quad (1.91)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L, \quad (1.92)$$

is not a well-posed problem. Hint: verify that

$$v_n(x, t) = \frac{1}{n} e^{n^2 t} \sin\left(\frac{n\pi x}{L}\right) \quad (1.93)$$

is a solution of the problem $v_t + v_{xx} = 0$, $v(0, t) = v(L, t) = 0$, $v(x, 0) = (1/n) \sin(n\pi x/L)$.

Chapter 2

Fourier series

2.1 Expansion of a function into a Fourier series

Fourier analysis deals with the expansion of functions into trigonometric series. Joseph Fourier (1768–1830), a French mathematician and engineer, introduced trigonometric series as a method for solving partial differential equations that model wave motion and heat transfer in matter. Today, Fourier analysis plays an important role in applications to various problems in physics and engineering. Moreover, a large part of modern analysis is the result of attempts to place Fourier series on a rigorous mathematical foundation. In this chapter we will present some results from the theory of Fourier series that we need for the further study of the subject.

Let $f: [-L, L] \rightarrow \mathbb{R}$ be a function defined on the symmetric interval $[-L, L]$. We are interested in whether f can be represented in the form of the trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]. \quad (2.1)$$

Here we need to answer the following questions.

- (1) Is it possible to represent the function by means of the series (2.1)?
- (2) How can we determine the coefficients a_n and b_n ?
- (3) What kind of convergence does the series (2.1) have: pointwise, uniform, or in some other sense?

We can answer the second question if we assume that the series (2.1) converges uniformly on $[-L, L]$. Here we use the fact that the functions

$$1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, \dots \quad (2.2)$$

satisfy the orthogonality relations

$$\int_{-L}^L 1 \cdot \cos\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^L 1 \cdot \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad (2.3)$$

$$\int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L \delta_{nm}, \quad (2.4)$$

$$\int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L \delta_{nm}, \quad (2.5)$$

$$\int_{-L}^L \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad n, m \geq 1 \quad (2.6)$$

where δ_{nm} is the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \quad (2.7)$$

The equalities (2.3)–(2.6) are easily proved by using trigonometric identities that convert products into sums of trigonometric functions. By integrating equation (2.1) we obtain

$$\int_{-L}^L f(x) dx = \frac{a_0}{2} \int_{-L}^L dx + \sum_{n=1}^{\infty} \left[a_n \int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) dx + b_n \int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) dx \right] = a_0 L \quad (2.8)$$

which implies

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx \quad (2.9)$$

(here we have interchanged the sum and the integral because, by assumption, the series (2.1) converges uniformly). Multiplying equation (2.1) by $\cos(m\pi x/L)$ and applying the orthogonality relation (2.6), we obtain

$$\begin{aligned} \int_{-L}^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx &= \frac{a_0}{2} \int_{-L}^L \cos\left(\frac{m\pi}{L}x\right) dx \\ &+ \sum_{n=1}^{\infty} \left[a_n \int_{-L}^L \cos\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx + b_n \int_{-L}^L \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{m\pi}{L}x\right) dx \right] \\ &= a_m L. \end{aligned} \quad (2.10)$$

Therefore,

$$a_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx, \quad m \geq 1. \quad (2.11)$$

In a similar way one proves that

$$b_m = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx, \quad m \geq 1. \quad (2.12)$$

Note that the constant term in the series (2.1) is defined as $a_0/2$ so that the expressions for a_0 and a_m can be written in the compact form

$$a_m = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx, \quad m \geq 0. \quad (2.13)$$

These preliminary considerations motivate the following definition.

Definition 2.1 (Fourier series) *The trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \quad (2.14)$$

where

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 0, 1, 2, \dots \quad (2.15)$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots \quad (2.16)$$

is called the *Fourier series* of the function f on the interval $[-L, L]$. The coefficients a_n and b_n are called the *Fourier coefficients*.

Before we study the important question of the convergence of the Fourier series, let us consider a few simple examples. We denote the Fourier series of the function f by \tilde{f} because, as we shall soon see, the sum of the series need not be equal to the value of the function at a given point.

When computing Fourier coefficients, it is useful to observe that the following rules hold:

(i) if $f: [-L, L] \rightarrow \mathbb{R}$ is an odd function, then

$$\int_{-L}^L f(x) dx = 0, \quad (2.17)$$

(ii) if $f: [-L, L] \rightarrow \mathbb{R}$ is an even function, then

$$\int_{-L}^L f(x)dx = 2 \int_0^L f(x)dx. \quad (2.18)$$

Example 2.1 Determine the Fourier series of the function $f(x) = x$ on the interval $[-L, L]$.

Solution. The function $x \cos(n\pi x/L)$ is odd, so

$$a_n = \frac{1}{L} \int_{-L}^L x \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad n \geq 0. \quad (2.19)$$

The coefficients b_n are given by

$$b_n = \frac{1}{L} \int_{-L}^L x \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \quad (2.20)$$

$$= \frac{2}{L} \left[-\frac{Lx}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right) \right] \Big|_0^L \quad (2.21)$$

$$= -\frac{2L}{n\pi} \cos(n\pi), \quad n \geq 1. \quad (2.22)$$

Since $\cos(n\pi) = (-1)^n$, we have

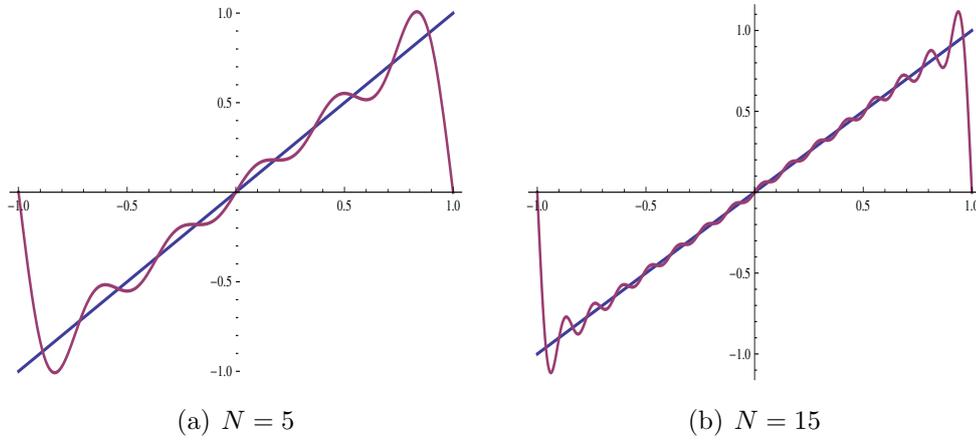
$$b_n = \frac{2L}{n\pi} (-1)^{n+1}. \quad (2.23)$$

Therefore, the Fourier series of the function $f(x) = x$ has the form

$$\begin{aligned} \tilde{f}(x) &= \frac{2L}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{2L}{\pi} \left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2} \sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{L}\right) - \dots \right] \end{aligned} \quad (2.24)$$

□

Figure 2.1 shows the partial sums of the series with $N = 5$ and $N = 15$ terms. Note that the sum of the Fourier series at the points $x = \pm L$ is not equal to the value of the function. At points $x \in (-L, L)$ the Fourier series converges to $f(x)$, as can be anticipated from Figure 2.1(b). However, the convergence is “slower” the closer the

Figure 2.1: Expansion of $f(x) = x$ into a Fourier series.

point x is to the endpoints $\pm L$ of the interval. It is interesting to note the following fact. For $L = \pi$, expression (2.24) takes the simpler form

$$\tilde{f}(x) = 2 \left(\sin(x) - \frac{1}{2} \sin(2x) + \frac{1}{3} \sin(3x) - \cdots \right). \quad (2.25)$$

If we substitute $x = \pi/2$, then $\tilde{f}(\pi/2) = \pi/2$, and we obtain Gregory's series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \quad (2.26)$$

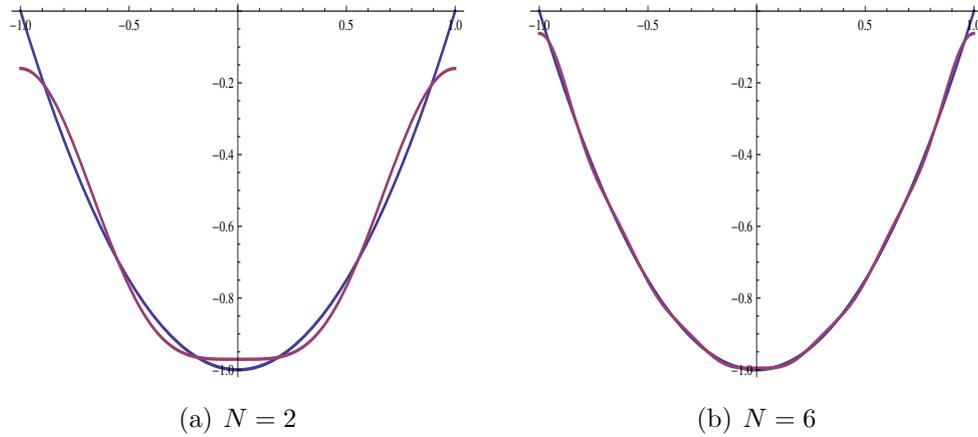
which can be used to compute approximations of the number π . ■

Example 2.2 Determine the Fourier series of the function $f(x) = x^2 - 1$ on the interval $[-1, 1]$.

Solution. The coefficients a_n are given by

$$a_0 = \int_{-1}^1 (x^2 - 1) dx = -\frac{4}{3}, \quad (2.27)$$

$$\begin{aligned} a_n &= \int_{-1}^1 (x^2 - 1) \cos(n\pi x) dx = \int_{-1}^1 x^2 \cos(n\pi x) dx - \int_{-1}^1 \cos(n\pi x) dx \\ &= \frac{1}{(n\pi)^3} \left[2n\pi x \cos(n\pi x) + ((n\pi x)^2 - 2) \sin(n\pi x) \right] \Big|_{-1}^1 - \frac{1}{n\pi} \sin(n\pi x) \Big|_{-1}^1 \\ &= \frac{4}{(n\pi)^2} \cos(n\pi) = \frac{4(-1)^n}{(n\pi)^2}, \quad n \geq 1. \end{aligned} \quad (2.28)$$

Figure 2.2: Expansion of $f(x) = x^2 - 1$ into a Fourier series.

The function $(x^2 - 1) \sin(n\pi x)$ is odd, so

$$b_n = \int_{-1}^1 (x^2 - 1) \sin(n\pi x) dx = 0, \quad n \geq 1. \quad (2.29)$$

Therefore, the Fourier series of the function f has the form

$$\begin{aligned} \tilde{f}(x) &= -\frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x) \\ &= -\frac{2}{3} + \frac{4}{\pi^2} \left(-\cos(\pi x) + \frac{1}{4} \cos(2\pi x) - \frac{1}{9} \cos(3\pi x) + \cdots \right) \end{aligned} \quad (2.30)$$

□

The partial sums of the series with $N = 2$ and $N = 6$ terms are shown in Figure 2.2. Note that this series converges faster than the series in the previous example, since a good approximation of the function is obtained with only $N = 6$ terms. Also, Figure 2.2(b) suggests that the series converges equally fast on the entire interval $[-1, 1]$. ■

The above examples show that expanding a function into a Fourier series is meaningful. From these examples we also observe that the sum of a Fourier series need not be equal to the value of the function at all points. Therefore, it is necessary to study more closely the conditions under which, and in what sense, the Fourier series converges to the given function.

2.2 Convergence of the Fourier series

A Fourier series is completely determined by the integral of the function f on the interval $[-L, L]$. If we change the value of the function at a single point, the value of the integral does not change, and therefore the Fourier series remains the same. Hence we cannot expect the Fourier series to converge to f at every point. This simple observation suggests that the problem of convergence of a Fourier series is quite subtle. It can be shown that there exists a continuous function whose Fourier series diverges at all rational points, while Kolmogorov showed around 1930 that there exists an integrable function (in the Lebesgue sense) whose Fourier series diverges everywhere. Here we restrict our attention to functions that are piecewise continuous and study the properties of the Fourier series of such functions. Such functions form a sufficiently large class for applying Fourier series to various problems.

Definition 2.2 *We say that a function f is piecewise continuous on $[a, b]$ if*

(i) *it is defined and continuous except possibly at finitely many points*

$$a \leq x_1 < x_2 < \dots < x_n \leq b,$$

(ii) *at the points of discontinuity $x_k \neq a, b$ the one-sided limits exist*

$$f(x_k^-) = \lim_{x \rightarrow x_k^-} f(x), \quad f(x_k^+) = \lim_{x \rightarrow x_k^+} f(x), \quad (2.31)$$

(iii) *at the endpoints the limits $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ exist.*

Note that a piecewise continuous function need not be defined at the points of discontinuity. If $f(x_k)$ is defined, then $f(x_k)$ need not be equal to the left-hand or the right-hand limit at x_k . At the points x_k the function f has a discontinuity of the first kind, and the difference

$$\beta_k = f(x_k^+) - f(x_k^-) \quad (2.32)$$

represents the jump of the function at the point of discontinuity. If $\beta_k = 0$, then the function has a removable discontinuity at the point x_k . One of the central results on the convergence of Fourier series concerns functions that are piecewise C^1 on the interval $[a, b]$.

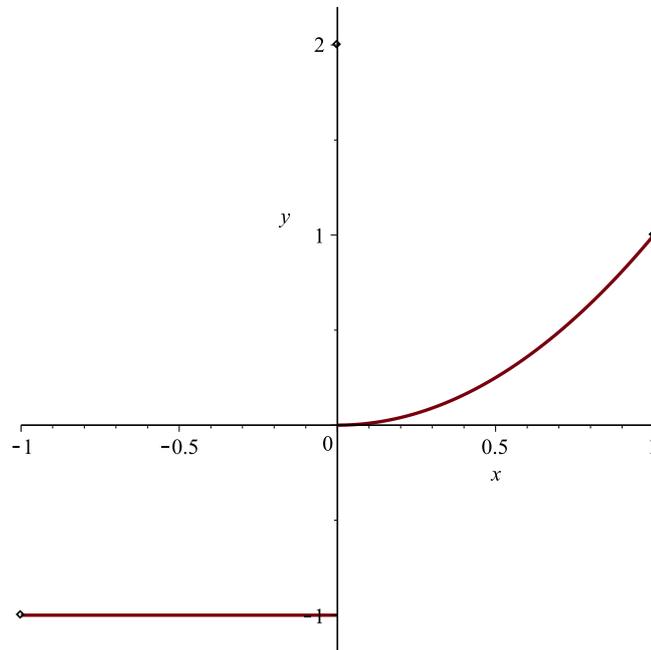


Figure 2.3: The function (2.33) is piecewise C^1 on $[-1, 1]$.

Definition 2.3 We say that a function f is piecewise C^1 on the interval $[a, b]$ if both f and f' are piecewise continuous on $[a, b]$.

According to this definition, f is piecewise C^1 on $[a, b]$ if the functions f and f' have at most finitely many points of discontinuity and at these points they have discontinuities of the first kind. For such a function we distinguish points of discontinuity of the first kind where $f(x_k^-) \neq f(x_k^+)$ and points where the function has a corner, that is, where $f(x_k^-) = f(x_k^+)$ but $f'(x_k^-) \neq f'(x_k^+)$. Let us illustrate these notions with the following example.

Example 2.3 Consider the function

$$f(x) = \begin{cases} -1, & -1 \leq x < 0, \\ 2, & x = 0, \\ x^2, & 0 < x \leq 1. \end{cases} \quad (2.33)$$

The function f is continuous on the open set $(-1, 0) \cup (0, 1)$. It has a discontinuity of the first kind at $x = 0$ since $f(0^-) = -1$ and $f(0^+) = 0$. At the endpoints we have

$f(-1^+) = -1$ and $f(1^-) = 1$. The derivative f' is given by

$$f'(x) = \begin{cases} 0, & -1 < x < 0, \\ 2x, & 0 < x < 1, \end{cases} \quad (2.34)$$

while it is not defined at the points $x = -1, 0$ and 1 . At the point $x = 0$ we have $f'(0^-) = f'(0^+) = 0$, and at the endpoints $f'(-1^+) = 0$ and $f'(1^-) = 2$. Thus f and f' satisfy the conditions of Definition 2.2, so f is piecewise C^1 on the interval $[-1, 1]$.

Roughly speaking, piecewise C^1 functions have finitely many discontinuities of the first kind and finitely many corners at which the function is not differentiable. In this case one can prove the following result, which we state without proof.

Theorem 2.1 (Dirichlet's theorem) *Let f be a piecewise C^1 function on $[-L, L]$ and let \tilde{f} be the Fourier series of the function f . Then the following holds:*

- (i) $\tilde{f}(x_0) = f(x_0)$ if f is continuous at the point $x_0 \in (-L, L)$,
- (ii) $\tilde{f}(x_0) = \frac{1}{2}[f(x_0^+) + f(x_0^-)]$ if f has a discontinuity at the point $x_0 \in (-L, L)$,
- (iii) $\tilde{f}(\pm L) = \frac{1}{2}[f(-L^+) + f(L^-)]$.

According to this theorem, if f has a discontinuity at x_0 , then $\tilde{f}(x_0)$ is equal to the average value of the one-sided limits at x_0 . The sum of the Fourier series at the endpoints $\pm L$ can also be interpreted as the average of the one-sided limits if the interval $[-L, L]$ is bent into a circle and the points $-L$ and L are identified.

Let us illustrate Dirichlet's theorem with the example of the step function

$$f(x) = \begin{cases} 0, & -1 \leq x < 0, \\ 1, & 0 < x \leq 1. \end{cases} \quad (2.35)$$

shown in Figure 2.4. The function f is clearly piecewise C^1 on the interval $[-1, 1]$.

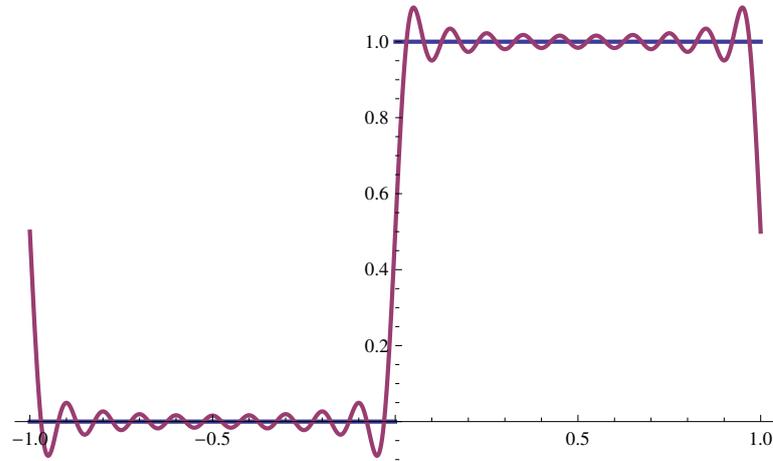


Figure 2.4: The Fourier series of the step function (2.35).

The Fourier coefficients are given by

$$a_0 = \int_0^1 dx = 1, \quad (2.36)$$

$$a_n = \int_0^1 \cos(n\pi x) dx = \frac{\sin(n\pi)}{n\pi} = 0, \quad n \geq 1, \quad (2.37)$$

$$b_n = \int_0^1 \sin(n\pi x) dx = \frac{1}{n\pi} (1 - (-1)^n) = \begin{cases} 0, & n = 2k, \\ \frac{2}{n\pi}, & n = 2k + 1. \end{cases} \quad (2.38)$$

Therefore,

$$\tilde{f}(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin((2k+1)\pi x). \quad (2.39)$$

The sum of the series (2.39) at $x = 0$ is equal to $\tilde{f}(0) = 1/2$. On the other hand, $f(0^-) = 0$ and $f(0^+) = 1$, so

$$\tilde{f}(0) = \frac{1}{2} [f(0^-) + f(0^+)] \quad (2.40)$$

in accordance with Dirichlet's theorem. Also, at the endpoints of the interval we have

$$\tilde{f}(\pm 1) = \frac{1}{2} [f(-1^+) + f(1^-)] \quad (2.41)$$

since $\tilde{f}(\pm 1) = 1/2$. Let us show that the series does not converge uniformly on $[-1, 1]$. The reason for this is the discontinuity of the function at the point $x = 0$. Let us

denote by $S_N(x)$ the N th partial sum of the Fourier series (2.39),

$$S_N(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^N \frac{1}{2k+1} \sin((2k+1)\pi x). \quad (2.42)$$

The function $S_N(x)$ is continuous and $S_N(0) = 1/2$, so for $\varepsilon = 1/4$ there exists $\delta > 0$ such that

$$|x - 0| < \delta \quad \Rightarrow \quad \left| S_N(x) - \frac{1}{2} \right| < \frac{1}{4}. \quad (2.43)$$

In particular, for $0 < x < \delta$, it follows from (2.43) that

$$\frac{1}{4} < S_N(x) < \frac{3}{4}. \quad (2.44)$$

On the other hand, $f(x) = 1$ for $0 < x < \delta$, so we have

$$|S_N(x) - f(x)| > f(x) - S_N(x) > 1 - \frac{3}{4} = \frac{1}{4}. \quad (2.45)$$

From this we conclude that

$$\frac{1}{4} < |S_N(x) - f(x)| \quad \text{for all } 0 < x < \delta, \quad N \geq 1, \quad (2.46)$$

which implies

$$\sup_{x \in [-1, 1]} |S_N(x) - f(x)| \geq \frac{1}{4} \quad \text{for every } N \geq 1. \quad (2.47)$$

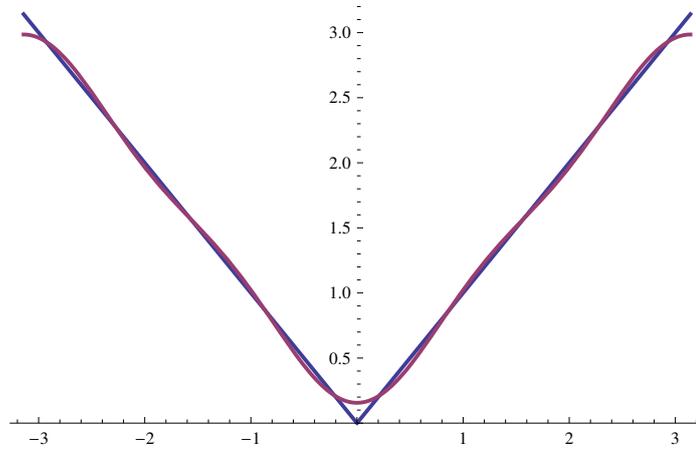
Therefore, the convergence of the series is not uniform on the set $[-1, 1]$.

Example 2.4 Determine the Fourier series of the function $f(x) = |x|$ on the interval $[-\pi, \pi]$.

Solution. This function is piecewise C^1 on $[-\pi, \pi]$ and continuous on $[-\pi, \pi]$. For the Fourier coefficients we obtain

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx = \pi, \quad (2.48)$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(nx) dx \\ &= \frac{2}{\pi} \left[\frac{x \sin(nx)}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{\sin(nx)}{n} dx \right] \\ &= \frac{2}{\pi} \frac{\cos(nx)}{n^2} \Big|_0^{\pi} = \frac{2}{\pi} \frac{(-1)^n - 1}{n^2}, \quad n \geq 1. \end{aligned} \quad (2.49)$$

Figure 2.5: The Fourier series of $f(x) = |x|$.

Since $f(x) = |x|$ is an even function, we have

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \sin(nx) dx = 0, \quad n \geq 1. \quad (2.50)$$

Therefore, the Fourier series has the form

$$\begin{aligned} \tilde{f}(x) &= \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(nx) \\ &= \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos((2n+1)x). \end{aligned} \quad (2.51)$$

(see Figure 2.5). Since f is continuous on $[-\pi, \pi]$, Dirichlet's theorem implies that $\tilde{f}(x) = f(x)$ for every $x \in [-\pi, \pi]$. Note that $\tilde{f}(0) = f(0) = 0$, so from relation (2.51) we obtain the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}. \quad (2.52)$$

□

2.2.1 Uniform convergence

In many applications of Fourier series it is desirable that the series converges uniformly. The importance of uniform convergence will become particularly clear when

solving partial differential equations by the method of separation of variables. Observe, for example, in the case of the step function (2.35) that continuity is a necessary condition for the uniform convergence of a Fourier series. In this chapter we will prove that, under some additional assumptions on the function f , the Fourier series converges uniformly to f . To prove this statement we need the following results.

Proposition 2.1 (Cauchy–Schwarz inequality) *Let z_i, w_i be complex numbers for $1 \leq i \leq n$. Then*

$$\left| \sum_{i=1}^n z_i \bar{w}_i \right| \leq \sqrt{\sum_{i=1}^n |z_i|^2} \sqrt{\sum_{i=1}^n |w_i|^2}. \quad (2.53)$$

Proof. Define $a = \sum_{i=1}^n z_i \bar{w}_i$. Then for every real number λ we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^n |z_i - \lambda a w_i|^2 = \sum_{i=1}^n (z_i - \lambda a w_i)(\bar{z}_i - \lambda \bar{a} \bar{w}_i) \\ &= \sum_{i=1}^n z_i \bar{z}_i - \lambda \sum_{i=1}^n a w_i \bar{z}_i - \lambda \sum_{i=1}^n z_i \bar{a} \bar{w}_i + \lambda^2 a \bar{a} \sum_{i=1}^n w_i \bar{w}_i \\ &= \sum_{i=1}^n |z_i|^2 - 2\lambda |a|^2 + \lambda^2 |a|^2 \sum_{i=1}^n |w_i|^2. \end{aligned} \quad (2.54)$$

The expression (2.54) is a quadratic polynomial $p(\lambda)$ in the variable λ such that $p(\lambda) \geq 0$. Therefore the discriminant of the polynomial

$$\Delta = (-2|a|^2)^2 - 4 \left(|a|^2 \sum_{i=1}^n |w_i|^2 \right) \left(\sum_{i=1}^n |z_i|^2 \right) \quad (2.55)$$

must satisfy $\Delta \leq 0$. If $a = 0$, then the condition is trivially satisfied. If $a \neq 0$, then by dividing by $|a|^2 \neq 0$ from the condition $\Delta \leq 0$ we obtain

$$|a|^2 \leq \sum_{i=1}^n |z_i|^2 \sum_{i=1}^n |w_i|^2, \quad (2.56)$$

that is,

$$\sum_{i=1}^n |z_i \bar{w}_i| \leq \sqrt{\sum_{i=1}^n |z_i|^2} \sqrt{\sum_{i=1}^n |w_i|^2}. \quad (2.57)$$

■

The Cauchy–Schwarz inequality is perhaps the most important inequality in mathematical analysis. Its generalization to unitary spaces is particularly important.

Theorem 2.2 (Bessel’s inequality) *Let $f: [-L, L] \rightarrow \mathbb{R}$ be a piecewise continuous function and let*

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 0, \quad (2.58)$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1, \quad (2.59)$$

be the Fourier coefficients of the function f . Then

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^L f(x)^2 dx. \quad (2.60)$$

Proof. Let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \quad (2.61)$$

be the N th partial sum of the Fourier series. Squaring gives

$$\begin{aligned} 0 &\leq \int_{-L}^L (f(x) - S_N(x))^2 dx \\ &= \int_{-L}^L (f(x))^2 dx - 2 \int_{-L}^L f(x) S_N(x) dx + \int_{-L}^L (S_N(x))^2 dx. \end{aligned} \quad (2.62)$$

From the definition of the Fourier coefficients we obtain by direct computation

$$\begin{aligned} \int_{-L}^L f(x) S_N(x) dx &= \int_{-L}^L f(x) \left[\frac{1}{2} a_0 + \sum_{n=1}^N \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \right] dx \\ &= \frac{1}{2} a_0 \int_{-L}^L f(x) dx + \sum_{n=1}^N \left[a_n \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx + b_n \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right] \\ &= L \left(\frac{1}{2} a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \right). \end{aligned} \quad (2.63)$$

Furthermore, the orthogonality relations (2.3)-(2.6) imply that

$$\begin{aligned}
 \int_{-L}^L (S_N(x))^2 dx &= \int_{-L}^L S_N(x) \left[\frac{1}{2}a_0 + \sum_{n=1}^N \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \right] dx \\
 &= \frac{1}{2}a_0 \int_{-L}^L S_N(x) dx + \sum_{n=1}^N \left[a_n \int_{-L}^L S_N(x) \cos\left(\frac{n\pi x}{L}\right) dx + b_n \int_{-L}^L S_N(x) \sin\left(\frac{n\pi x}{L}\right) dx \right] \\
 &= L \left(\frac{1}{2}a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \right). \tag{2.64}
 \end{aligned}$$

Substituting expressions (2.63) and (2.64) into (2.62) yields

$$0 \leq \int_{-L}^L f(x)^2 dx - L \left(\frac{1}{2}a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \right) \tag{2.65}$$

that is,

$$\frac{1}{2}a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx. \tag{2.66}$$

Since inequality (2.66) holds for every $N \geq 1$, we conclude that

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{L} \int_{-L}^L (f(x))^2 dx. \tag{2.67}$$

■

We say that a function is square-integrable on the interval $[a, b]$ if

$$\int_a^b (f(x))^2 dx < \infty. \tag{2.68}$$

For square-integrable functions it can be shown that the *Parseval identity* holds:

$$\frac{1}{2}a_0^2 + \sum_{n=1}^N (a_n^2 + b_n^2) = \frac{1}{L} \int_{-L}^L (f(x))^2 dx. \tag{2.69}$$

Piecewise continuous functions are square-integrable on a finite interval, so such functions satisfy equality (2.69). The proof of Parseval's identity goes beyond the scope of this text since it requires knowledge of the theory of Hilbert spaces. This identity is often useful for summing series of real numbers, as illustrated by the following example.

Example 2.5 Determine the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution. Recall that the function $f(x) = x$, $x \in [-L, L]$, in Example 2.1 has Fourier coefficients $a_n = 0$ for every $n \geq 0$ and

$$b_n = \frac{2L}{n\pi}(-1)^{n+1}, \quad n \geq 1. \quad (2.70)$$

Therefore, from relation (2.69) we obtain

$$\sum_{n=1}^{\infty} \left(\frac{2L}{n\pi} \right)^2 = \frac{1}{L} \int_{-L}^L x^2 dx = \frac{2}{3}L^2 \quad (2.71)$$

which implies

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad (2.72)$$

□

A direct consequence of Bessel's inequality is

Theorem 2.3 (Riemann–Lebesgue lemma) *Let $f: [-L, L] \rightarrow \mathbb{R}$ be a piecewise continuous function. Then the Fourier coefficients satisfy*

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{L} \lim_{n \rightarrow \infty} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad (2.73)$$

$$\lim_{n \rightarrow \infty} b_n = \frac{1}{L} \lim_{n \rightarrow \infty} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0. \quad (2.74)$$

Proof. Bessel's inequality implies that $\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$ converges, so from the necessary condition for the convergence of a series it follows that $\lim_{n \rightarrow \infty} a_n^2 = \lim_{n \rightarrow \infty} b_n^2 = 0$. Therefore $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 0$. ■

We can now prove our main result.

Theorem 2.4 (Uniform convergence theorem) *Let f be a continuous and piecewise C^1 function on $[-L, L]$ such that $f(-L) = f(L)$. Then the Fourier series converges uniformly to f on the set $[-L, L]$.*

Proof. Let

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] \quad (2.75)$$

be the N th partial sum of the Fourier series of the function f . By Dirichlet's theorem we have

$$\lim_{N \rightarrow \infty} S_N(x) = f(x) \quad \text{for every } x \in [-L, L] \quad (2.76)$$

since f is continuous and piecewise C^1 on $[-L, L]$. Now

$$\begin{aligned} |f(x) - S_N(x)| &= \left| \sum_{n=N+1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right] \right| \\ &\leq \sum_{n=N+1}^{\infty} \left| a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right| \\ &\leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|) \quad \text{for every } x \in [-L, L]. \end{aligned} \quad (2.77)$$

If we show that the series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge, then uniform convergence will follow from inequality (2.77). The coefficients a_n can be written using integration by parts in the form

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx \quad (2.78)$$

$$= \frac{1}{L} \left[f(x) \frac{L}{n\pi} \sin\left(\frac{n\pi}{L}x\right) \Big|_{-L}^L - \int_{-L}^L f'(x) \frac{L}{n\pi} \sin\left(\frac{n\pi}{L}x\right) dx \right] \quad (2.79)$$

$$= -\frac{L}{n\pi} \frac{1}{L} \int_{-L}^L f'(x) \sin\left(\frac{n\pi}{L}x\right) dx = -\frac{L}{n\pi} A_n, \quad n \geq 1, \quad (2.80)$$

where

$$A_n = \frac{1}{L} \int_{-L}^L f'(x) \sin\left(\frac{n\pi}{L}x\right) dx. \quad (2.81)$$

Similarly, using the condition $f(-L) = f(L)$ we obtain

$$b_n = \frac{L}{n\pi} \frac{1}{L} \int_{-L}^L f'(x) \cos\left(\frac{n\pi}{L}x\right) dx = \frac{L}{n\pi} B_n, \quad n \geq 1 \quad (2.82)$$

where

$$B_n = \frac{1}{L} \int_{-L}^L f'(x) \cos\left(\frac{n\pi}{L}x\right) dx. \quad (2.83)$$

Thus the Fourier coefficients for f and f' satisfy

$$|a_n| = \frac{L}{\pi n} |A_n| \quad \text{and} \quad |b_n| = \frac{L}{\pi n} |B_n|, \quad n \geq 1. \quad (2.84)$$

Let us now show that the series $\sum_{n=1}^{\infty} \frac{1}{n} |A_n|$ and $\sum_{n=1}^{\infty} \frac{1}{n} |B_n|$ converge. Note that

$$B_0 = \frac{1}{L} \int_{-L}^L f'(x) dx = \frac{1}{L} (f(L) - f(-L)) = 0. \quad (2.85)$$

Therefore Bessel's inequality for the function f' implies

$$\sum_{n=1}^{\infty} (A_n^2 + B_n^2) \leq \frac{1}{L} \int_{-L}^L (f'(x))^2 dx < \infty. \quad (2.86)$$

By the Cauchy–Schwarz inequality, for every $N \in \mathbb{N}$ we have

$$\sum_{n=1}^N \frac{1}{n} |A_n| \leq \sqrt{\sum_{n=1}^N \frac{1}{n^2}} \sqrt{\sum_{n=1}^N A_n^2} \leq \frac{\pi^2}{6} \sqrt{\sum_{n=1}^{\infty} A_n^2} \quad (2.87)$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Similarly we obtain

$$\sum_{n=1}^N \frac{1}{n} |B_n| \leq \frac{\pi}{\sqrt{6}} \sqrt{\sum_{n=1}^{\infty} B_n^2} \quad (2.88)$$

for every $N \geq 1$. Inequalities (2.87) and (2.88) imply that the series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge. Therefore from (2.77) it follows that

$$\lim_{N \rightarrow \infty} \sup_{x \in [-L, L]} |f(x) - S_N(x)| \leq \lim_{N \rightarrow \infty} \sum_{n=N+1}^{\infty} (|a_n| + |b_n|) = 0. \quad (2.89)$$

This proves that the Fourier series converges uniformly to f on $[-L, L]$. ■

Roughly speaking, the Uniform Convergence Theorem holds for continuous functions that have at most a finite number of cusps in the interval $[-L, L]$ where the first derivative does not exist. An example of such a function is $f(x) = |x|$, which has a cusp at the point $x = 0$.

A particularly important aspect of the theory of Fourier series is the relationship between the smoothness of a function and the rate at which the Fourier coefficients tend to zero. In the proof of Theorem 2.4, the assumption that f' is piecewise continuous implies that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, which means that $|a_n|$ and $|b_n|$ tend to zero faster than $1/n$. Knowing an upper bound for the coefficients $|a_n|$ and $|b_n|$ allows us to estimate how many terms of the Fourier series are required to achieve a given accuracy in approximating the function.

Proposition 2.2 *Let $f \in C^2[-L, L]$ be such that $f(-L) = f(L)$ and $f'(-L) = f'(L)$. Let $M = \max_{x \in [-L, L]} |f''(x)|$. Then the Fourier coefficients satisfy the upper bounds*

$$|a_n| = \left| \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \right| \leq \frac{2L^2 M}{\pi^2 n^2}, \quad (2.90)$$

$$|b_n| = \left| \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \right| \leq \frac{2L^2 M}{\pi^2 n^2}, \quad n \geq 1. \quad (2.91)$$

Proof. Integrating by parts we obtain

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{L} \left[\frac{L}{n\pi} f(x) \sin\left(\frac{n\pi x}{L}\right) \Big|_{x=-L}^{x=L} - \frac{L}{n\pi} \int_{-L}^L f'(x) \sin\left(\frac{n\pi x}{L}\right) dx \right] \\ &= -\frac{1}{n\pi} \int_{-L}^L f'(x) \sin\left(\frac{n\pi x}{L}\right) dx. \end{aligned} \quad (2.92)$$

From the condition $f'(-L) = f'(L)$ it follows that

$$\begin{aligned} \int_{-L}^L f'(x) \sin\left(\frac{n\pi x}{L}\right) dx &= -\frac{L}{n\pi} f'(x) \cos\left(\frac{n\pi x}{L}\right) \Big|_{x=-L}^{x=L} + \frac{L}{n\pi} \int_{-L}^L f''(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{L}{n\pi} \int_{-L}^L f''(x) \cos\left(\frac{n\pi x}{L}\right) dx \end{aligned} \quad (2.93)$$

so by substituting (2.93) into (2.92) we obtain

$$a_n = -\frac{L}{n^2 \pi^2} \int_{-L}^L f''(x) \cos\left(\frac{n\pi x}{L}\right) dx. \quad (2.94)$$

Similarly, it can be shown that the condition $f(-L) = f(L)$ implies

$$b_n = -\frac{L}{n^2 \pi^2} \int_{-L}^L f''(x) \sin\left(\frac{n\pi x}{L}\right) dx. \quad (2.95)$$

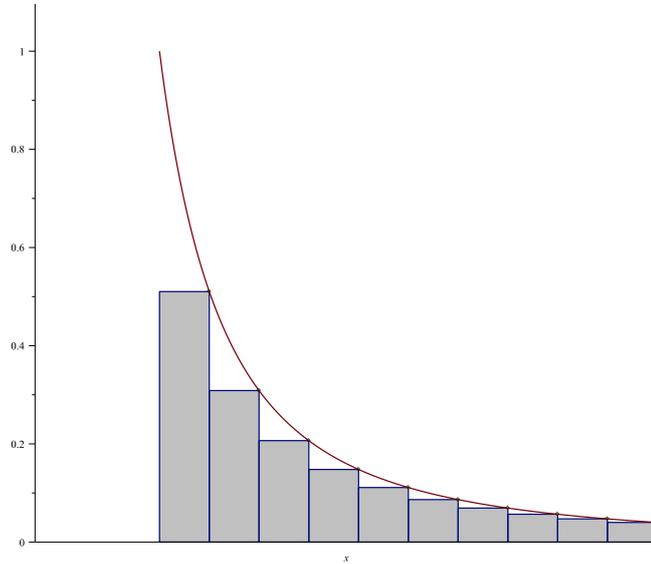


Figure 2.6: The sum of the series $\sum_{n=N+1}^{\infty} \frac{1}{n^2}$

From equations (2.94) and (2.95) we conclude that the upper bounds for $|a_n|$ and $|b_n|$ are given by

$$|a_n| \leq \frac{L}{n^2\pi^2} \int_{-L}^L \left| f''(x) \cos\left(\frac{n\pi x}{L}\right) \right| dx \leq \frac{2L^2 M}{n^2\pi^2}, \quad (2.96)$$

$$|b_n| \leq \frac{L}{n^2\pi^2} \int_{-L}^L \left| f''(x) \sin\left(\frac{n\pi x}{L}\right) \right| dx \leq \frac{2L^2 M}{n^2\pi^2} \quad (2.97)$$

where $M = \max_{x \in [-L, L]} |f''(x)|$. ■

These estimates for the Fourier coefficients are useful when a_n and b_n are not explicitly known, for example when they are computed numerically. In that case, relations (2.96) and (2.97) provide a rough estimate of how many terms of the Fourier series are needed to approximate the function within a prescribed accuracy. The difference between the function $f(x)$ and the N th partial sum of the Fourier series $S_N(x)$ is

bounded by

$$\begin{aligned} |f(x) - S_N(x)| &\leq \left| \sum_{n=N+1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right) \right| \\ &\leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|) = \frac{4L^2M}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} \end{aligned} \quad (2.98)$$

for all $x \in [-L, L]$. Note that the sum of the series $\sum_{n=N+1}^{\infty} \frac{1}{n^2}$ is smaller than the area under the curve $y = \frac{1}{x^2}$ for $N \leq x < \infty$ (see Figure 2.6). Therefore

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{dx}{x^2} = \frac{1}{N} \quad (2.99)$$

so from (2.98) we obtain the estimate

$$\sup_{x \in [-L, L]} |f(x) - S_N(x)| \leq \frac{4L^2M}{\pi^2 N}. \quad (2.100)$$

If we want the approximation error to be less than $\epsilon > 0$, then N should be chosen such that

$$N > \frac{4L^2M}{\pi^2 \epsilon}. \quad (2.101)$$

This method for estimating the number N is called the integral method because in relation (2.99) the sum of the series is majorized by an integral. Much sharper estimates for N can be obtained if the Fourier coefficients are computed explicitly and then the integral method is applied to bound the series. Let us illustrate this procedure with the following example.

Example 2.6 Estimate how many terms of the Fourier series are required to approximate the function $f(x) = x^3 - x$, $x \in [-1, 1]$, with an error smaller than $\epsilon = 0.01$.

Solution. The function $f(x) = x^3 - x$ is odd, so $a_n = 0$ for every $n \geq 0$. The coefficients b_n are given by

$$b_n = \int_{-1}^1 (x^3 - x) \sin(n\pi x) dx = (-1)^n \frac{12}{(n\pi)^3}. \quad (2.102)$$

Hence the Fourier series is

$$\tilde{f}(x) = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(n\pi x). \quad (2.103)$$

By Theorem 2.4, the series (2.103) converges uniformly to f and we have

$$\left| f(x) - \frac{12}{\pi^3} \sum_{n=1}^N \frac{(-1)^n}{n^3} \sin(n\pi x) \right| = \left| \frac{12}{\pi^3} \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n^3} \sin(n\pi x) \right| \leq \frac{12}{\pi^3} \sum_{n=N+1}^{\infty} \frac{1}{n^3} \quad (2.104)$$

for every $x \in [-1, 1]$. The sum $\sum_{n=N+1}^{\infty} 1/n^3$ is smaller than the area under the curve $y = 1/x^3$ for $N \leq x < \infty$, therefore

$$\sum_{n=N+1}^{\infty} \frac{1}{n^3} \leq \int_N^{\infty} \frac{1}{x^3} dx = \frac{1}{2N^2}. \quad (2.105)$$

From (2.104) we obtain the estimate

$$\left| f(x) - \frac{12}{\pi^3} \sum_{n=1}^N \frac{(-1)^n}{n^3} \sin(n\pi x) \right| \leq \frac{6}{\pi^3 N^2}. \quad (2.106)$$

If we want the approximation error to be less than $\epsilon = 0.01$, then N should satisfy $6/(\pi^3 N^2) < 0.01$, that is

$$N > \sqrt{\frac{6}{\pi^3 \cdot 0.01}} \approx 4.4. \quad (2.107)$$

Thus it is sufficient to take $N = 5$ terms of the series. For comparison, let us determine N from relation (2.101). The maximal value of the second derivative is

$$M = \max_{-1 \leq x \leq 1} |6x| = 6, \quad (2.108)$$

so condition (2.101) implies

$$N > \frac{4 \cdot 6}{\pi^2 \cdot 0.01} \approx 243.17, \quad (2.109)$$

that is $N = 244$. Clearly the first estimate is much better than the estimate obtained from relation (2.101). \square

Chapter 3

Quasi-linear first-order equations

In this chapter we consider the general solution of the quasi-linear first-order equation

$$P(x, y, u)u_x + Q(x, y, u)u_y = R(x, y, u). \quad (3.1)$$

Assume that P , Q , and R are C^1 functions on a domain $\Omega \subseteq \mathbb{R}^3$ which do not vanish simultaneously at any point $(x, y, u) \in \Omega$. A solution of equation (3.1) is a function $u = u(x, y)$ of class C^1 on a domain $\Omega_0 \subseteq \mathbb{R}^2$ such that $(x, y, u(x, y)) \in \Omega$ for every point $(x, y) \in \Omega_0$. In other words, the functions $P(x, y, u(x, y))$, $Q(x, y, u(x, y))$, and $R(x, y, u(x, y))$ are well defined on the domain Ω_0 . The solution $u = u(x, y)$ can be regarded as a level surface of the function $f(x, y, u) = u(x, y) - u$,

$$S = \left\{ (x, y, u) \mid f(x, y, u) = 0 \right\}, \quad (3.2)$$

which we call the *integral surface* of equation (3.1). The vector $\nabla f = u_x \vec{e}_1 + u_y \vec{e}_2 - \vec{e}_3$ is orthogonal to the surface S at every point $(x, y, u) \in S$ where $\nabla f \neq 0$. Observe that equation (3.1) can be written as the scalar product of the vectors

$$(P\vec{e}_1 + Q\vec{e}_2 + R\vec{e}_3) \cdot (u_x \vec{e}_1 + u_y \vec{e}_2 - \vec{e}_3) = 0 \quad (3.3)$$

which implies that the vector $\vec{F} = P\vec{e}_1 + Q\vec{e}_2 + R\vec{e}_3$ is orthogonal to ∇f whenever $\nabla f \neq 0$. We conclude that \vec{F} must be a tangent vector to the integral surface S at every point where $\nabla f \neq 0$. The line determined by the vector \vec{F} is called the Monge axis and plays a crucial role in solving equation (3.1).

Definition 3.1 We say that $\gamma \subset S$ is a characteristic curve if at every point $(x, y, u) \in \gamma$ the tangent vector to γ equals $\vec{F}(x, y, u) = P(x, y, u)\vec{e}_1 + Q(x, y, u)\vec{e}_2 + R(x, y, u)\vec{e}_3$.

If γ is defined by the parametric equations

$$\gamma: \quad x = x(t), \quad y = y(t), \quad u = u(t), \quad t \in I, \quad (3.4)$$

then

$$x'(t) = P(x, y, u), \quad y'(t) = Q(x, y, u), \quad u'(t) = R(x, y, u). \quad (3.5)$$

Equations (3.5) are called the *characteristic equations* of the quasi-linear equation (3.1). The characteristic equations can also be written in the nonparametric form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}. \quad (3.6)$$

Finding the general solution of equation (3.1) reduces to solving the characteristic equations (3.5), or equivalently (3.6). This method is called the *Lagrange method of characteristics*. The solutions of equations (3.6) can in general be written in the form $\phi(x, y, u) = C$ for some $C \in \mathbb{R}$. We say that $\phi(x, y, u) = C_1$ and $\psi(x, y, u) = C_2$ are functionally independent solutions if

$$\nabla\phi \times \nabla\psi \neq 0. \quad (3.7)$$

Theorem 3.1 (Method of characteristics) *Let $\phi(x, y, u) = C_1$ and $\psi(x, y, u) = C_2$ be two functionally independent solutions of the characteristic equations (3.6) in a domain $\Omega \subseteq \mathbb{R}^3$ where the functions P , Q , and R are defined. The general solution of equation (3.1) is given by $f(\phi, \psi) = 0$ where f is an arbitrary C^1 function.*

Proof. Let $\gamma \subset S$ be a characteristic curve with parametrization $x = x(t)$, $y = y(t)$, $u = u(t)$, $t \in I$. Then $\phi(x(t), y(t), u(t)) = C_1$ and therefore

$$\frac{d\phi}{dt} = \phi_x x'(t) + \phi_y y'(t) + \phi_u u'(t) = P\phi_x + Q\phi_y + R\phi_u = 0. \quad (3.8)$$

Similarly, from $\psi(x(t), y(t), u(t)) = C_2$ we obtain

$$\frac{d\psi}{dt} = P\psi_x + Q\psi_y + R\psi_u = 0. \quad (3.9)$$

Since ϕ and ψ are functionally independent, we have

$$\nabla\phi \times \nabla\psi = \vec{i} \begin{vmatrix} \phi_y & \phi_u \\ \psi_y & \psi_u \end{vmatrix} - \vec{j} \begin{vmatrix} \phi_x & \phi_u \\ \psi_x & \psi_u \end{vmatrix} + \vec{k} \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} \neq 0. \quad (3.10)$$

Without loss of generality assume that $\phi_x\psi_y - \phi_y\psi_x \neq 0$. Then using equations (3.8) and (3.9) the functions P and Q can be expressed in terms of R ,

$$P = -R \frac{\begin{vmatrix} \phi_u & \phi_y \\ \psi_u & \psi_y \end{vmatrix}}{\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}}, \quad Q = -R \frac{\begin{vmatrix} \phi_x & \phi_u \\ \psi_x & \psi_u \end{vmatrix}}{\begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}}. \quad (3.11)$$

On the other hand, differentiating the expression $f(\phi, \psi) = 0$ gives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \phi} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} \right) + \frac{\partial f}{\partial \psi} \left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial x} \right) = 0, \quad (3.12)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \phi} \left(\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} \right) + \frac{\partial f}{\partial \psi} \left(\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial u} \frac{\partial u}{\partial y} \right) = 0, \quad (3.13)$$

where we have taken into account that the variable u depends on x and y . The system of equations (3.12)–(3.13) has nontrivial solutions for $\partial f/\partial \phi$ and $\partial f/\partial \psi$ only if

$$\begin{vmatrix} \phi_x + \phi_u u_x & \psi_x + \psi_u u_x \\ \phi_y + \phi_u u_y & \psi_y + \psi_u u_y \end{vmatrix} = 0. \quad (3.14)$$

Condition (3.14) can be written in the form

$$\begin{vmatrix} \phi_u & \phi_y \\ \psi_u & \psi_y \end{vmatrix} u_x + \begin{vmatrix} \phi_x & \phi_u \\ \psi_x & \psi_u \end{vmatrix} u_y = - \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}. \quad (3.15)$$

Now from equations (3.11) and (3.15) it follows that u satisfies the differential equation

$$Pu_x + Qu_y = R. \quad (3.16)$$

■

Observe that the same conclusion follows if we assume that any of the determinants in equation (3.10) is nonzero.

Example 3.1 Determine the general solution of the quasi-linear equation

$$x^2 u_x + y^2 u_y = (x + y)u. \quad (3.17)$$

Solution. The characteristic curves are determined by the equations

$$x'(t) = x^2, \quad y'(t) = y^2, \quad u'(t) = (x + y)u. \quad (3.18)$$

Integrating the first two equations in (3.18) we obtain

$$x = -\frac{1}{t + C_1}, \quad y = -\frac{1}{t + C_2}, \quad (3.19)$$

where C_1 and C_2 are constants of integration. Substituting (3.19) into the third equation in (3.18) we see that the function u satisfies

$$u'(t) = -\left(\frac{1}{t + C_1} + \frac{1}{t + C_2}\right)u. \quad (3.20)$$

Integrating equation (3.20) yields

$$u(t) = \frac{K_1}{(t + C_1)(t + C_2)} \quad (3.21)$$

for some constant K_1 . From equations (3.19) and (3.21) we can now determine functions ϕ and ψ that remain constant along the characteristic curves. Observe that

$$\frac{1}{t + C_1} \frac{1}{t + C_2} = xy, \quad (3.22)$$

so from equation (3.21) it follows that

$$\frac{u}{xy} = K_1. \quad (3.23)$$

Furthermore, equation (3.19) implies

$$\frac{1}{x} - \frac{1}{y} = C_2 - C_1 = K_2. \quad (3.24)$$

If we define the functions

$$\phi(x, y, u) = \frac{u}{xy}, \quad \psi(x, y, u) = \frac{y - x}{xy}, \quad (3.25)$$

then from (3.23) and (3.24) it follows that $\phi = K_1$ and $\psi = K_2$ along the characteristic curves. Hence the general solution is given by

$$f\left(\frac{u}{xy}, \frac{y - x}{xy}\right) = 0 \quad (3.26)$$

where f is an arbitrary C^1 function. \square

Chapter 4

Second-order equations in two independent variables

In this chapter we study linear second-order equations in two independent variables. These equations can be classified into three types: hyperbolic, parabolic, and elliptic equations. The fundamental equations of mathematical physics, namely the wave equation, the diffusion equation, and Laplace's equation, are important examples of these types of equations. Solutions of the same type have similar qualitative properties, and each type of equation can be reduced by a change of variables to a so-called canonical form which is simpler to study.

The general linear second-order equation in two independent variables has the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (4.1)$$

where u, A, B, C, D, E, F, G are functions of the variables x, y in a given domain $\Omega \subseteq \mathbb{R}^2$. We assume that the functions A, B , and C do not vanish simultaneously in Ω and that $u \in C^2(\Omega)$. Equation (4.1) can be written in operator form as $L[u] = G$, where

$$L = A \frac{\partial^2}{\partial x^2} + 2B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} + F. \quad (4.2)$$

The operator

$$L_0 = A \frac{\partial^2}{\partial x^2} + 2B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2} \quad (4.3)$$

is called the principal part of the operator L . Associated with the operator L_0 is the discriminant

$$\Delta(x, y) = B^2(x, y) - A(x, y)C(x, y). \quad (4.4)$$

We will show that the sign of the discriminant Δ is invariant under a regular change of variables, that is, it does not depend on the coordinate system in which we consider the equation. This suggests that second-order equations can be classified according to the sign of the discriminant Δ .

Definition 4.1 *We say that equation (4.1) is*

(a) *hyperbolic at the point (x, y) if $\Delta(x, y) > 0$,*

(b) *parabolic at the point (x, y) if $\Delta(x, y) = 0$,*

(c) *elliptic at the point (x, y) if $\Delta(x, y) < 0$.*

If equation (4.1) is hyperbolic (parabolic, elliptic) at every point of the domain Ω , then we say that it is hyperbolic (parabolic, elliptic) in Ω . The classification of equations into these types is motivated by conic sections, since the equation

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0 \quad (4.5)$$

represents a hyperbola, parabola, or ellipse depending on whether the discriminant $\Delta = B^2 - AC$ is positive, zero, or negative.

The wave equation

$$u_{tt} - c^2 u_{xx} = 0 \quad (4.6)$$

is hyperbolic in \mathbb{R}^2 because $\Delta = c^2 > 0$ ($A = -c^2, B = 0, C = 1$). The heat equation

$$u_t - k u_{xx} = 0, \quad k > 0, \quad (4.7)$$

is parabolic in \mathbb{R}^2 because $\Delta = 0$ ($A = -k, B = C = 0$), while Laplace's equation

$$u_{xx} + u_{yy} = 0 \quad (4.8)$$

is elliptic in \mathbb{R}^2 because $\Delta = -1 < 0$ ($A = C = 1, B = 0$). Tricomi's equation

$$y u_{xx} + u_{yy} = 0 \quad (4.9)$$

has discriminant $\Delta = -y$ ($A = y, B = 0, C = 1$). The equation is elliptic in the half-plane $y > 0$, hyperbolic in the half-plane $y < 0$, and parabolic on the line $y = 0$.

We are interested in how equation (4.1) transforms under the introduction of new variables

$$\alpha = \alpha(x, y), \quad \beta = \beta(x, y), \quad (x, y) \in \Omega. \quad (4.10)$$

We will assume that $\alpha, \beta \in C^2(\Omega)$ and that the Jacobian of the transformation

$$J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} = \alpha_x \beta_y - \alpha_y \beta_x \neq 0 \quad \forall (x, y) \in \Omega. \quad (4.11)$$

In that case there exists an inverse transformation

$$x = x(\alpha, \beta), \quad y = y(\alpha, \beta), \quad (4.12)$$

and the transformed function $w(\alpha, \beta) = u(x(\alpha, \beta), y(\alpha, \beta))$ is of class C^2 .

Lemma 4.1 *Let*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (4.13)$$

be a linear second-order equation. Let $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ be a regular change of variables. Then the sign of the discriminant $\Delta = B^2 - AC$ is invariant under the transformation $(x, y) \mapsto (\alpha, \beta)$.

Lemma 4.1 shows that the sign of the discriminant $\Delta(x, y)$ is an intrinsic property of equation (4.13), since the functions $u(x, y)$ and $w(\alpha, \beta)$ satisfy an equation of the same type.

Proof. The functions u and w are related by $u(x, y) = w(\alpha(x, y), \beta(x, y))$. Applying the chain rule we obtain

$$u_x = w_\alpha \alpha_x + w_\beta \beta_x, \quad (4.14)$$

$$u_y = w_\alpha \alpha_y + w_\beta \beta_y, \quad (4.15)$$

$$u_{xx} = w_{\alpha\alpha} \alpha_x^2 + 2w_{\alpha\beta} \alpha_x \beta_x + w_{\beta\beta} \beta_x^2 + w_\alpha \alpha_{xx} + w_\beta \beta_{xx}, \quad (4.16)$$

$$u_{xy} = w_{\alpha\alpha} \alpha_x \alpha_y + w_{\alpha\beta} (\alpha_x \beta_y + \alpha_y \beta_x) + w_{\beta\beta} \beta_x \beta_y + w_\alpha \alpha_{xy} + w_\beta \beta_{xy}, \quad (4.17)$$

$$u_{yy} = w_{\alpha\alpha} \alpha_y^2 + 2w_{\alpha\beta} \alpha_y \beta_y + w_{\beta\beta} \beta_y^2 + w_\alpha \alpha_{yy} + w_\beta \beta_{yy}. \quad (4.18)$$

Substituting these expressions into (4.13), it follows that the function w satisfies the transformed equation

$$\bar{L}[w] \equiv \bar{A}w_{\alpha\alpha} + 2\bar{B}w_{\alpha\beta} + \bar{C}w_{\beta\beta} + \bar{D}w_{\alpha} + \bar{E}w_{\beta} + \bar{F}w = \bar{G} \quad (4.19)$$

where

$$\bar{A}(\alpha, \beta) = A\alpha_x^2 + 2B\alpha_x\alpha_y + C\alpha_y^2, \quad (4.20)$$

$$\bar{B}(\alpha, \beta) = A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + C\alpha_y\beta_y, \quad (4.21)$$

$$\bar{C}(\alpha, \beta) = A\beta_x^2 + 2B\beta_x\beta_y + C\beta_y^2. \quad (4.22)$$

We do not need to compute the remaining coefficients explicitly because the type of the equation depends only on the coefficients A , B , and C . Equations (4.20)–(4.22) can be written in matrix form as

$$\begin{pmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{C} \end{pmatrix} = \begin{pmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} \alpha_x & \beta_x \\ \alpha_y & \beta_y \end{pmatrix}. \quad (4.23)$$

Taking determinants on the left-hand and right-hand sides of equation (4.23), we obtain

$$\bar{A}\bar{C} - \bar{B}^2 = (AC - B^2)(\alpha_x\beta_y - \alpha_y\beta_x)^2, \quad (4.24)$$

that is,

$$\bar{\Delta} = \Delta J^2 \quad (4.25)$$

where $J = \alpha_x\beta_y - \alpha_y\beta_x$ is the Jacobian of the transformation. Since $(x, y) \mapsto (\alpha, \beta)$ is a regular transformation, we have $J \neq 0$, and therefore we conclude that the discriminants $\bar{\Delta}$ and Δ have the same sign, or else they are both zero. This implies that the functions u and w satisfy an equation of the same type. ■

By introducing new variables, equation (4.1) can be transformed into a simpler form, the so-called *canonical form*. Studying canonical forms provides a clearer insight into the general properties of equation (4.1). If the solution $w(\alpha, \beta)$ of the canonical form of the equation is known, then the solution of the original equation is given by $u(x, y) = w(\alpha(x, y), \beta(x, y))$. We distinguish the following canonical forms of second-order equations.

Definition 4.2 (1) The canonical form of a **hyperbolic** equation is

$$u_{xy} + L_1[u] = G \quad (4.26)$$

where L_1 is a first-order differential operator. This canonical form is equivalent to

$$w_{\alpha\alpha} - w_{\beta\beta} + L_1[w] = G \quad (4.27)$$

where the variables α, β are defined by the transformation $\alpha = x + y, \beta = x - y$.

(2) The canonical form of a **parabolic** equation is

$$u_{xx} + L_1[u] = G. \quad (4.28)$$

(3) The canonical form of an **elliptic** equation is

$$u_{xx} + u_{yy} + L_1[u] = G. \quad (4.29)$$

Observe that the fundamental equations of mathematical physics—the wave equation, the diffusion equation, and Laplace’s equation—are precisely the principal parts of hyperbolic, parabolic, and elliptic equations, respectively. In the next chapter we study the changes of variables that transform equations into their canonical forms.

4.1 Canonical form of hyperbolic equations

Theorem 4.1 *Let*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (4.30)$$

be a hyperbolic equation in a domain $\Omega \subseteq \mathbb{R}^2$. Then there exist variables $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ in which equation (4.30) takes the canonical form

$$w_{\alpha\beta} + L_1[w] = \bar{G} \quad (4.31)$$

where $w(\alpha, \beta) = u(x(\alpha, \beta), y(\alpha, \beta))$ and L_1 is a first-order differential operator.

Proof. If $A = C = 0$, then $\Delta = B^2 > 0$, and in this case the equation reduces to the canonical form simply by dividing by $2B \neq 0$. Now suppose without loss of generality that $A \neq 0$ in the domain Ω . If $C \neq 0$, we can interchange the roles of x and y . The equation will be reduced to the canonical form if we determine variables $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ such that

$$\bar{A} = A\alpha_x^2 + 2B\alpha_x\alpha_y + C\alpha_y^2 = 0, \quad (4.32)$$

$$\bar{C} = A\beta_x^2 + 2B\beta_x\beta_y + C\beta_y^2 = 0. \quad (4.33)$$

The quadratic equation

$$A\lambda^2 + 2B\lambda + C = 0 \quad (4.34)$$

has two distinct real solutions $\lambda_1(x, y)$ and $\lambda_2(x, y)$ in Ω ,

$$\lambda_{1,2} = \frac{-B \pm \sqrt{B^2 - AC}}{A}, \quad (4.35)$$

since $\Delta = B^2 - AC > 0$.

Let $\alpha(x, y)$ and $\beta(x, y)$ be nontrivial solutions of the first-order equations

$$\alpha_x = \lambda_1(x, y)\alpha_y, \quad (4.36)$$

$$\beta_x = \lambda_2(x, y)\beta_y. \quad (4.37)$$

Equations (4.36) and (4.37) are called the *characteristic equations*. Substituting these relations into the expressions for the coefficients \bar{A} and \bar{C} yields

$$\bar{A} = (A\lambda_1^2 + 2B\lambda_1 + C)\alpha_y^2 = 0, \quad (4.38)$$

$$\bar{C} = (A\lambda_2^2 + 2B\lambda_2 + C)\beta_y^2 = 0. \quad (4.39)$$

Thus equation (4.30) in the new variables takes the form

$$2\bar{B}w_{\alpha\beta} + L_1[w] = \bar{G} \quad (4.40)$$

where L_1 is a first-order differential operator. According to Lemma 4.1 the transformed discriminant satisfies $\bar{\Delta} = \bar{B}^2 > 0$, therefore equation (4.40) can be divided by $2\bar{B} \neq 0$, which gives the canonical form (4.31).

It remains to verify that the transformation $\alpha = \alpha(x, y)$, $\beta = \beta(x, y)$ is regular. Substituting equations (4.36) and (4.37) into the Jacobian of the transformation we obtain

$$J = \begin{vmatrix} \lambda_1 \alpha_y & \alpha_y \\ \lambda_2 \beta_y & \beta_y \end{vmatrix} = (\lambda_1 - \lambda_2) \alpha_y \beta_y. \quad (4.41)$$

Here $\lambda_1 - \lambda_2 \neq 0$ because λ_1 and λ_2 are distinct solutions of equation (4.34). Since α is a nontrivial solution of the characteristic equation, we must have $\alpha_y \neq 0$, otherwise from (4.36) it would follow that $\alpha_x = \alpha_y = 0$, hence $\alpha = \text{const}$. Similarly we conclude that $\beta_y \neq 0$. Therefore $J \neq 0$, and nontrivial solutions of the characteristic equations determine a regular change of variables. ■

If λ_1 and λ_2 are constants, then the characteristic equations have simple solutions $\alpha(x, y) = a(\lambda_1 x + y)$ and $\beta(x, y) = b(\lambda_2 x + y)$ where $a, b \in \mathbb{R}$ are arbitrary.

Example 4.1 Determine the canonical form and the general solution of the equation

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2. \quad (4.42)$$

Solution. We have $A = 4$, $B = 5/2$, and $C = 1$, and $\Delta = 9/4 > 0$, hence the equation is hyperbolic in \mathbb{R}^2 . The quadratic equation $A\lambda^2 + 2B\lambda + C = 0$ has two real roots $\lambda_1 = -1/4$ and $\lambda_2 = -1$. The characteristic equations are

$$\alpha_x = -\frac{1}{4}\alpha_y, \quad \beta_x = -\beta_y, \quad (4.43)$$

which yields

$$\alpha = -\frac{1}{4}x + y, \quad \beta = -x + y. \quad (4.44)$$

Let $w(\alpha, \beta) = u(x(\alpha, \beta), y(\alpha, \beta))$. Then

$$u_x = -\frac{1}{4}w_\alpha - w_\beta, \quad (4.45)$$

$$u_y = w_\alpha + w_\beta, \quad (4.46)$$

$$u_{xx} = \frac{1}{16}w_{\alpha\alpha} + \frac{1}{2}w_{\alpha\beta} + w_{\beta\beta}, \quad (4.47)$$

$$u_{yy} = w_{\alpha\alpha} + 2w_{\alpha\beta} + w_{\beta\beta}, \quad (4.48)$$

$$u_{xy} = -\frac{1}{4}w_{\alpha\alpha} - \frac{5}{4}w_{\alpha\beta} - w_{\beta\beta}. \quad (4.49)$$

Substituting expressions (4.45)–(4.49) into equation (4.42) we obtain the canonical form

$$w_{\alpha\beta} = \frac{1}{3}w_{\alpha} - \frac{8}{9}. \quad (4.50)$$

The function $v = w_{\alpha}$ satisfies the first-order equation

$$v_{\beta} - \frac{1}{3}v = -\frac{8}{9}.$$

The general solution is

$$v(\alpha, \beta) = B(\alpha)e^{\frac{1}{3}\beta} + \frac{8}{3},$$

where the constant of integration B depends on the variable α . Hence

$$w(\alpha, \beta) = \int v(\alpha, \beta) d\alpha = \int B(\alpha) d\alpha e^{\frac{1}{3}\beta} + \frac{8}{3}\alpha + C. \quad (4.51)$$

Therefore the general solution of the original equation (4.42) has the form

$$u(x, y) = w\left(-\frac{1}{4}x + y, -x + y\right) = f\left(-\frac{1}{4}x + y\right)e^{\frac{1}{3}(-x+y)} - \frac{2}{3}x + \frac{8}{3}y + C \quad (4.52)$$

where f is an arbitrary C^2 function. Note that if we introduce the variables $\xi = \alpha + \beta$ and $\eta = \alpha - \beta$ and define the function $\bar{w}(\xi, \eta) = w(\alpha(\xi, \eta), \beta(\xi, \eta))$, then the equivalent canonical form of equation (4.42) is given by

$$\bar{w}_{\xi\xi} - \bar{w}_{\eta\eta} = \frac{1}{3}\bar{w}_{\xi} + \frac{1}{3}\bar{w}_{\eta} - \frac{8}{9}. \quad (4.53)$$

□

4.2 Canonical form of parabolic equations

Theorem 4.2 *Let*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (4.54)$$

be a parabolic equation in a domain $\Omega \subseteq \mathbb{R}^2$. Then there exist variables $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ in which equation (4.54) takes the canonical form

$$w_{\alpha\alpha} + L_1[w] = \bar{G} \quad (4.55)$$

where $w(\alpha, \beta) = u(x(\alpha, \beta), y(\alpha, \beta))$ and L_1 is a first-order differential operator.

Proof. Since A , B , and C are not all zero, the assumption $\Delta = B^2 - AC = 0$ implies that $A \neq 0$ or $C \neq 0$. Without loss of generality assume that $A \neq 0$ in Ω . If $C \neq 0$, the proof is modified in the obvious way. According to Lemma 4.1, it is enough to determine variables $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ such that

$$\bar{B} = A\alpha_x \beta_x + B(\alpha_x \beta_y + \alpha_y \beta_x) + C\alpha_y \beta_y = 0, \quad (4.56)$$

$$\bar{C} = A\beta_x^2 + 2B\beta_x \beta_y + C\beta_y^2 = 0. \quad (4.57)$$

Because of the condition $\Delta = B^2 - AC = 0$, the equation $A\lambda^2 + 2B\lambda + C = 0$ has a single real solution in Ω ,

$$\lambda(x, y) = -\frac{B(x, y)}{A(x, y)}. \quad (4.58)$$

Let β be a nontrivial solution of the characteristic equation

$$\beta_x = \lambda(x, y)\beta_y. \quad (4.59)$$

Substituting equation (4.59) into (4.56) and (4.57) we obtain

$$\bar{B} = (A\lambda + B)\alpha_x \beta_y + (B\lambda + C)\alpha_y \beta_y = \frac{1}{A}(AC - B^2)\alpha_y \beta_y = 0, \quad (4.60)$$

$$\bar{C} = (A\lambda^2 + 2B\lambda + C)\beta_y^2 = 0. \quad (4.61)$$

For $\alpha(x, y)$ we may take any function for which the Jacobian of the transformation $(x, y) \mapsto (\alpha, \beta)$ does not vanish. If we choose $\alpha = x$, then

$$J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} = \beta_y \neq 0 \quad (4.62)$$

because β is a nontrivial solution of the characteristic equation. Now for the coefficients we have

$$\bar{A} = A\alpha_x^2 + 2B\alpha_x \alpha_y + C\alpha_y^2 = A \neq 0 \quad (4.63)$$

and $\bar{B} = \bar{C} = 0$, so the transformed equation is given by $\bar{A}w_{\alpha\alpha} + L_1[w] = \bar{G}$. Dividing by $\bar{A} \neq 0$ we obtain the canonical form (4.55). ■

Example 4.2 Determine the canonical form of the equation

$$x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} + xu_x + yu_y = 0 \quad (4.64)$$

and find the general solution of the equation in the half-plane $\Omega = \{(x, y) \mid x > 0\}$.

Solution. The equation is parabolic in \mathbb{R}^2 because $A = x^2$, $B = -xy$, and $C = y^2$, which implies $\Delta = B^2 - AC = (-xy)^2 - x^2y^2 = 0$. By the previous theorem we need to determine a solution of the characteristic equation

$$\beta_x = \lambda\beta_y \quad \text{where} \quad \lambda = -\frac{B}{A} = \frac{y}{x}. \quad (4.65)$$

One solution is given by $\beta = xy$, hence the change of variables is $\alpha = x$, $\beta = xy$. The transformation is regular in the half-plane Ω because the Jacobian of the transformation is $J = \beta_y = x > 0$ in Ω . Define $w(\alpha, \beta) = u(x(\alpha, \beta), y(\alpha, \beta))$. Then

$$u_x = w_\alpha + w_\beta \frac{\beta}{\alpha}, \quad (4.66)$$

$$u_y = w_\beta \alpha, \quad (4.67)$$

$$u_{xx} = w_{\alpha\alpha} + 2w_{\alpha\beta} \frac{\beta}{\alpha} + w_{\beta\beta} \left(\frac{\beta}{\alpha}\right)^2, \quad (4.68)$$

$$u_{xy} = w_{\alpha\beta} \alpha + w_{\beta\beta} \beta + w_\beta, \quad (4.69)$$

$$u_{yy} = w_{\beta\beta} \alpha^2. \quad (4.70)$$

Substituting these expressions into equation (4.64) we obtain

$$\alpha^2 w_{\alpha\alpha} + \alpha w_\alpha = 0. \quad (4.71)$$

By integrating the equation we find

$$w_\alpha = \frac{1}{\alpha} f(\beta), \quad (4.72)$$

so the function w is given by

$$w = \int w_\alpha d\alpha = f(\beta) \ln |\alpha| + g(\beta) \quad (4.73)$$

where f and g are arbitrary C^2 functions. Therefore, the general solution of equation (4.64) in the half-plane $x > 0$ is the function

$$u(x, y) = f(xy) \ln(x) + g(xy). \quad (4.74)$$

□

4.3 Canonical form of elliptic equations

Determining variables in which an elliptic equation takes its canonical form is, in general, more complicated than in the hyperbolic or parabolic case (see [P.R. Garabedian, Partial Differential Equations, John Wiley and Sons, New York, 1964]). However, if the coefficients of the second derivatives are constant, then the procedure for reducing the equation to canonical form is similar to the hyperbolic case.

Theorem 4.3 *Let*

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \quad (4.75)$$

be an elliptic equation in a domain $\Omega \subset \mathbb{R}^2$. Then there exist variables $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ in which equation (4.75) takes the canonical form

$$w_{\alpha\alpha} + w_{\beta\beta} + L_1[w] = \bar{G} \quad (4.76)$$

where $w(\alpha, \beta) = u(x(\alpha, \beta), y(\alpha, \beta))$ and L_1 is a first-order differential operator.

Proof. We give the proof in the case when the coefficients A , B , and C are constant. From the condition $\Delta = B^2 - AC < 0$ it follows that $A \neq 0$ and $C \neq 0$. The coefficients of the principal part of the equation transform according to the rule

$$\bar{A} = A\alpha_x^2 + 2B\alpha_x\alpha_y + C\alpha_y^2, \quad (4.77)$$

$$\bar{B} = A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + C\alpha_y\beta_y, \quad (4.78)$$

$$\bar{C} = A\beta_x^2 + 2B\beta_x\beta_y + C\beta_y^2. \quad (4.79)$$

We wish to determine variables $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$ such that $\bar{A} = \bar{C} \neq 0$ and $\bar{B} = 0$. In that case α and β satisfy the equations $\bar{A} - \bar{C} = 0$ and $\bar{B} = 0$, that is,

$$A(\alpha_x^2 - \beta_x^2) + 2B(\alpha_x\alpha_y - \beta_x\beta_y) + C(\alpha_y^2 - \beta_y^2) = 0, \quad (4.80)$$

$$A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + C\alpha_y\beta_y = 0. \quad (4.81)$$

The system of equations (4.80)–(4.81) is equivalent to

$$A\phi_x^2 + 2B\phi_x\phi_y + C\phi_y^2 = 0 \quad (4.82)$$

where ϕ is the complex-valued function $\phi = \alpha + i\beta$. The equation $A\lambda^2 + 2B\lambda + C = 0$ has two complex conjugate solutions

$$\lambda_1 = \frac{-B + i\sqrt{AC - B^2}}{A}, \quad \lambda_2 = \frac{-B - i\sqrt{AC - B^2}}{A} \quad (4.83)$$

because $AC - B^2 > 0$. Let ϕ be a nontrivial solution of the characteristic equation

$$\phi_x = \lambda_1 \phi_y. \quad (4.84)$$

Then

$$A\phi_x^2 + 2B\phi_x\phi_y + C\phi_y^2 = (A\lambda_1^2 + 2B\lambda_1 + C)\phi_y^2 = 0 \quad (4.85)$$

which means that for this choice of the function ϕ we have $\bar{A} = \bar{C}$ and $\bar{B} = 0$. Define $a = -B/A$ and $b = \sqrt{AC - B^2}/A$ so that $\lambda_1 = a + ib$. Equation (4.84) has the solution $\phi = \lambda_1 x + y$, from which it follows that the desired transformation is given by

$$\alpha = \operatorname{Re}(\phi) = ax + y, \quad \beta = \operatorname{Im}(\phi) = bx. \quad (4.86)$$

The transformation is regular in \mathbb{R}^2 because

$$J = \begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix} = \begin{vmatrix} a & 1 \\ b & 0 \end{vmatrix} = -b \neq 0. \quad (4.87)$$

For the coefficients \bar{A} and \bar{C} we have

$$\bar{A} = Aa^2 + 2Ba + C = A \frac{AC - B^2}{A^2} = Ab^2, \quad \bar{C} = Ab^2, \quad (4.88)$$

therefore the transformed equation is given by $Ab^2 w_{\alpha\alpha} + Ab^2 w_{\beta\beta} + L_1[w] = \bar{G}$. Dividing by $Ab^2 \neq 0$ we obtain the canonical form (4.76). ■

Let us note that the elliptic equation can also be reduced to canonical form if for ϕ we choose $\phi_x = \lambda_2 \phi_y$. In that case the equations differ only in the differential operator L_1 , but the solution $u(x, y)$ has the same form.

Example 4.3 Determine the canonical form of the equation

$$u_{xx} + u_{xy} + u_{yy} + u_x = 0. \quad (4.89)$$

Solution. The coefficients of the principal part of the equation are $A = 1$, $B = 1/2$, and $C = 1$. The equation is elliptic in \mathbb{R}^2 because

$$\Delta = B^2 - AC = -\frac{3}{4} < 0. \quad (4.90)$$

The solutions of the equation $A\lambda^2 + 2B\lambda + C = 0$ are

$$\lambda_{1,2} = \frac{-1 \pm i\sqrt{3}}{2}, \quad (4.91)$$

so according to equation (4.86) the transformed variables have the form

$$\alpha = -\frac{1}{2}x + y, \quad \beta = \frac{\sqrt{3}}{2}x. \quad (4.92)$$

From this we obtain

$$u_x = -\frac{1}{2}w_\alpha + \frac{\sqrt{3}}{2}w_\beta, \quad (4.93)$$

$$u_{xx} = \frac{1}{4}w_{\alpha\alpha} - \frac{\sqrt{3}}{2}w_{\alpha\beta} + \frac{3}{4}w_{\beta\beta}, \quad (4.94)$$

$$u_{xy} = -\frac{1}{2}w_{\alpha\alpha} + \frac{\sqrt{3}}{2}w_{\alpha\beta}, \quad (4.95)$$

$$u_{yy} = w_{\beta\beta}. \quad (4.96)$$

Substituting expressions (4.93)–(4.95) into equation (4.89) we find

$$\frac{3}{4}w_{\alpha\alpha} + \frac{3}{4}w_{\beta\beta} - \frac{1}{2}w_\alpha + \frac{\sqrt{3}}{2}w_\beta = 0, \quad (4.97)$$

that is,

$$w_{\alpha\alpha} + w_{\beta\beta} - \frac{2}{3}w_\alpha + \frac{2}{\sqrt{3}}w_\beta = 0. \quad (4.98)$$

If instead of λ_1 we choose the solution $\lambda_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, then the transformed variables are given by

$$\alpha = -\frac{1}{2}x + y, \quad \beta = -\frac{\sqrt{3}}{2}x. \quad (4.99)$$

In that case the corresponding canonical form is

$$w_{\alpha\alpha} + w_{\beta\beta} - \frac{2}{3}w_\alpha - \frac{2}{\sqrt{3}}w_\beta = 0. \quad (4.100)$$

Observe that the principal part of the canonical form is unique, but the remaining part depends on the choice of the root λ_1 or λ_2 . \square

Problems

1. Let u be a solution of the equation

$$u_{xx} + 2u_{xy} + u_{yy} = 0. \quad (4.101)$$

Write the equation in the coordinates $s = x$, $t = x - y$ and determine the general solution.

2. Write the equation

$$u_{xx} - 2u_{xy} + 5u_{yy} = 0 \quad (4.102)$$

in the coordinates $s = x + y$, $t = 2x$, and determine the general solution of the equation.

3. Classify the following equations:

$$x^2 u_{xy} - y u_{yy} + u_x - 4u = 0, \quad (4.103)$$

$$xy u_{xx} + 4u_{xy} - (x^2 + y^2)u_{yy} - u = 0. \quad (4.104)$$

4. For each of the following equations determine the regions in the plane where the equations are hyperbolic, parabolic, or elliptic:

$$2u_{xx} + 4u_{xy} + 3u_{yy} - u = 0, \quad (4.105)$$

$$u_{xx} + 2xu_{xy} + u_{yy} + \sin(xy)u = 5, \quad (4.106)$$

$$yu_{xx} - 2u_{xy} + e^x u_{yy} + x^2 u_x - u = 0. \quad (4.107)$$

5. Reduce the following equations to canonical form:

$$c^2 u_{xx} - u_{yy} = 0, \quad (4.108)$$

$$2u_{xx} + u_{xy} + y u_{yy} = 0, \quad y > 1, \quad (4.109)$$

$$x^2 u_{xx} - 2xy u_{xy} + y^2 u_{yy} = 0, \quad (4.110)$$

$$x u_{xx} - 4u_{xy} = 0, \quad x > 0. \quad (4.111)$$

Chapter 5

The Heat Equation

The heat equation, or diffusion equation, describes the distribution of temperature in a thermally conductive body. It is the most important example of a partial differential equation of parabolic type. The first part of this chapter is devoted to the study of qualitative properties of solutions of the equation, including the maximum principle and the stability of solutions with respect to the initial and boundary conditions. In the second part of the chapter we construct solutions of the diffusion equation using the method of separation of variables and Fourier series.

5.1 Maximum Principle and Uniqueness of Solutions

Consider the heat equation on a finite interval:

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0. \quad (5.1)$$

The function $u(x, t)$ describes the temperature in a thin, homogeneous, thermally conductive rod at the point x and at time t . We assume that the rod is insulated except possibly at the endpoints $x = 0$ and $x = L$, and that there are no internal sources that heat or cool the rod. The constant $k > 0$ depends on the material of the body and is called the *thermal conductivity*.

Physical intuition suggests that the temperature distribution is determined if the initial temperature $u(x, 0)$ and the temperature at the endpoints of the rod, $u(0, t)$

and $u(L, t)$, are known. This leads us to consider the initial–boundary value problem with *Dirichlet* conditions

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (5.2)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L, \quad (5.3)$$

$$u(0, t) = a(t), \quad u(L, t) = b(t), \quad t \geq 0. \quad (5.4)$$

In the following we assume that the functions f , a , and b are continuous. Compatibility of the initial and boundary conditions implies that $f(0) = a(0)$ and $f(L) = b(0)$.

If, instead of the temperature, its gradient is prescribed at the endpoints of the rod, then the function u satisfies *Neumann* conditions

$$u_x(0, t) = a(t), \quad u_x(L, t) = b(t), \quad t \geq 0. \quad (5.5)$$

In this case the functions f , a , and b satisfy the compatibility conditions $f'(0) = a(0)$ and $f'(L) = b(0)$.

The function u is defined on the domain (see Figure 5.1)

$$\Omega = \{(x, t) \mid 0 < x < L, \quad t > 0\}, \quad (5.6)$$

and the boundary and initial conditions are prescribed on the boundary of the domain

$$\partial\Omega = \{(x, 0) \mid 0 \leq x \leq L\} \cup \{(0, t) \mid t \geq 0\} \cup \{(L, t) \mid t \geq 0\}. \quad (5.7)$$

Our task is to determine a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies equation (5.2) together with the initial and boundary conditions (5.3)–(5.4).

Theorem 5.1 (Uniqueness of the Solution) *If u_1 and u_2 are C^2 solutions of the problem (5.2)–(5.4), then $u_1 = u_2$.*

Proof. Let $w = u_1 - u_2$. The function w satisfies the equation

$$w_t - kw_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (5.8)$$

$$w(x, 0) = 0, \quad 0 \leq x \leq L, \quad (5.9)$$

$$w(0, t) = w(L, t) = 0, \quad t \geq 0. \quad (5.10)$$

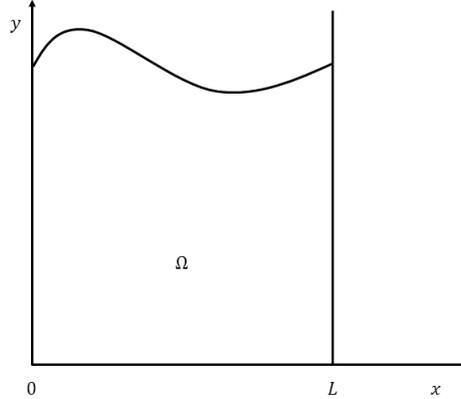


Figure 5.1: Domain for the heat equation.

Define the auxiliary function

$$J(t) = \frac{1}{2k} \int_0^L w^2(x, t) dx. \quad (5.11)$$

Since $(w^2)_t = 2w w_t$ is a continuous function, by Leibniz's rule we may differentiate $J(t)$ under the integral sign, obtaining

$$J'(t) = \frac{1}{2k} \int_0^L \frac{\partial}{\partial t} w^2 dx = \frac{1}{k} \int_0^L w w_t dx = \int_0^L w w_{xx} dx, \quad (5.12)$$

because $w_t = k w_{xx}$. Integration by parts yields

$$\int_0^L w w_{xx} dx = w w_x \Big|_{x=0}^{x=L} - \int_0^L w_x^2 dx = - \int_0^L w_x^2 dx, \quad (5.13)$$

where we used the fact that $w(0, t) = w(L, t) = 0$. Therefore,

$$J'(t) = - \int_0^L w_x^2 dx \leq 0, \quad (5.14)$$

which implies that $J(t)$ is a nonincreasing function. Furthermore, the initial condition $w(x, 0) = 0$ implies $J(0) = 0$. The conditions $J(0) = 0$ and $J'(t) \leq 0$ together imply

that $J(t) \leq 0$ for every $t \geq 0$. However, from the definition (5.11) we have $J(t) \geq 0$, which implies that

$$J(t) = \frac{1}{2k} \int_0^L w^2 dx = 0 \quad (5.15)$$

for every $t \geq 0$. Since $w^2 \geq 0$, this implies $w = 0$, and hence $u_1 = u_2$. ■

Note that by the same argument the Neumann problem also has a unique solution, since the boundary term in equation (5.13) vanishes when $w_x(0, t) = w_x(L, t) = 0$.

We now prove an interesting result which states that on every finite time interval $[0, T]$ the solution of the homogeneous equation (5.1) attains its maximum on the parabolic boundary of the rectangle $D = [0, L] \times [0, T]$,

$$\partial_p D = \{(0, t) \mid 0 \leq t \leq T\} \cup \{(x, 0) \mid 0 \leq x \leq L\} \cup \{(L, t) \mid 0 \leq t \leq T\}. \quad (5.16)$$

The parabolic boundary $\partial_p D$ is the union of the sides $x = 0$, $t = 0$, and $x = L$.

Theorem 5.2 (Maximum Principle) *Let u be a C^2 solution of the equation*

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0. \quad (5.17)$$

Let $T > 0$ and let D be the closed rectangle $[0, L] \times [0, T]$. Then the function u attains its maximum on D on the parabolic boundary $\partial_p D$, that is,

$$\max_{(x,t) \in D} u(x, t) = u(x_0, t_0) \quad (5.18)$$

for some point $(x_0, t_0) \in \partial_p D$.

Proof. Let $M = \max_{(x,t) \in D} u(x, t)$. Since u is continuous on D , there exists a point $(x_0, t_0) \in D$ such that $M = u(x_0, t_0)$. Suppose that the function does not attain its maximum on the parabolic boundary, that is, $(x_0, t_0) \in D \setminus \partial_p D$. Then

$$\max_{(x,t) \in \partial_p D} u(x, t) = M - \varepsilon \quad (5.19)$$

for some $\varepsilon > 0$. Introduce the auxiliary function

$$v(x, t) = u(x, t) + \frac{\varepsilon}{2L^2}(x - x_0)^2. \quad (5.20)$$

For points on the parabolic boundary we have $|x - x_0| < L$, and therefore equality (5.19) implies

$$v(x, t) \leq u(x, t) + \frac{\varepsilon}{2} \leq M - \frac{\varepsilon}{2}, \quad (x, t) \in \partial_p D. \quad (5.21)$$

On the other hand,

$$v(x_0, t_0) = u(x_0, t_0) = M > M - \frac{\varepsilon}{2}, \quad (5.22)$$

so we conclude that $\max_{(x,t) \in D} v(x, t)$ is not attained on the parabolic boundary $\partial_p D$. Therefore

$$\max_{(x,t) \in D} v(x, t) = v(x_1, t_1) \quad \text{at some point } (x_1, t_1) \in D \setminus \partial_p D. \quad (5.23)$$

At the point (x_1, t_1) the function v satisfies the necessary conditions for a maximum:

$$v_t(x_1, t_1) = 0, \quad v_{xx}(x_1, t_1) \leq 0 \quad \text{if } 0 < t_1 < T, \quad (5.24)$$

or

$$v_t(x_1, t_1) \geq 0, \quad v_{xx}(x_1, t_1) \leq 0 \quad \text{if } t_1 = T. \quad (5.25)$$

In both cases we have

$$v_t(x_1, t_1) - kv_{xx}(x_1, t_1) \geq 0. \quad (5.26)$$

However, from the definition of v we obtain

$$v_t(x_1, t_1) - kv_{xx}(x_1, t_1) = u_t(x_1, t_1) - ku_{xx}(x_1, t_1) - \frac{k\varepsilon}{L^2} < 0 \quad (5.27)$$

since $u_t(x_1, t_1) - ku_{xx}(x_1, t_1) = 0$ and $k\varepsilon > 0$, which contradicts relation (5.26). Therefore we conclude that the function u attains its maximum on D at some point of the parabolic boundary $\partial_p D$. ■

The physical interpretation of this principle is the following. The temperature in the interior of the rod (at a point $x \in (0, L)$) at any time $0 \leq t \leq T$ is smaller than the maximum initial temperature or the maximum temperature on the boundary of the rod. In geometric terms, the surface $u = u(x, t)$ attains its maximal height on one of the sides $x = 0$, $x = L$, or $t = 0$ of the rectangle $[0, L] \times [0, T]$.

Corollary 5.1 (Minimum Principle) *If the function u satisfies the assumptions of Theorem 5.2, then u attains its minimum at some point of the parabolic boundary $\partial_p D$.*

Proof. The function $w = -u$ satisfies the heat equation in Theorem 5.2, and therefore w attains its maximum at some point $(x_0, t_0) \in \partial_p D$. This implies that $u = -w$ attains its minimum at (x_0, t_0) . ■

The maximum and minimum principles imply the stability of solutions of the heat equation. More precisely, on every finite time interval $[0, T]$, a small change in the initial or boundary conditions results in a small change in the solution. This result is important because in applied problems the initial and boundary conditions are not always known exactly.

Theorem 5.3 (Stability of Solutions) *Let u_1 and u_2 be C^2 solutions of the initial-boundary value problems*

$$\frac{\partial u_i}{\partial t} - k \frac{\partial^2 u_i}{\partial x^2} = 0, \quad 0 < x < L, \quad t > 0, \quad (5.28)$$

$$u_i(x, 0) = f_i(x), \quad 0 \leq x \leq L, \quad (5.29)$$

$$u_i(0, t) = a_i(t), \quad u_i(L, t) = b_i(t), \quad t \geq 0 \quad (5.30)$$

for $i = 1, 2$. Let $T > 0$ and let $D = [0, L] \times [0, T]$. If

$$\max_{0 \leq x \leq L} |f_1(x) - f_2(x)| < \varepsilon, \quad (5.31)$$

$$\max_{0 \leq t \leq T} |a_1(t) - a_2(t)| < \varepsilon, \quad \max_{0 \leq t \leq T} |b_1(t) - b_2(t)| < \varepsilon \quad (5.32)$$

for some $\varepsilon > 0$, then

$$\max_{(x,t) \in D} |u_1(x, t) - u_2(x, t)| < \varepsilon. \quad (5.33)$$

Proof. The function $v = u_1 - u_2$ satisfies the equation $v_t - kv_{xx} = 0$, and on the parabolic boundary of D we have

$$|v(x, 0)| = |f_1(x) - f_2(x)| < \varepsilon, \quad 0 \leq x \leq L, \quad (5.34)$$

$$|v(0, t)| = |a_1(t) - a_2(t)| < \varepsilon, \quad 0 \leq t \leq T, \quad (5.35)$$

$$|v(L, t)| = |b_1(t) - b_2(t)| < \varepsilon, \quad 0 \leq t \leq T. \quad (5.36)$$

This implies that

$$|v(x, t)| < \varepsilon, \quad (x, t) \in \partial_p D, \quad (5.37)$$

that is,

$$-\varepsilon < v(x, t) < \varepsilon, \quad (x, t) \in \partial_p D. \quad (5.38)$$

By the maximum and minimum principles we have

$$-\varepsilon < \min_{(x,t) \in D} v(x, t) \quad \text{and} \quad \max_{(x,t) \in D} v(x, t) < \varepsilon, \quad (5.39)$$

which implies

$$\max_{(x,t) \in D} |u_1(x, t) - u_2(x, t)| = \max_{(x,t) \in D} |v(x, t)| < \varepsilon. \quad (5.40)$$

■

5.2 Separation of Variables for the Homogeneous Equation

In this section we determine the solution of the heat equation using the method of separation of variables and Fourier series. By this method the solution is obtained in the form of a series of eigenfunctions associated with the corresponding Sturm–Liouville problem. We will show that, under certain assumptions on the initial conditions, the resulting series converges and represents a classical solution of the equation.

To begin with, consider the boundary value problem with Dirichlet conditions.

Dirichlet Boundary Conditions

Let us determine the solution of the heat equation

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (5.41)$$

with the initial and boundary conditions

$$u(x, 0) = f(x), \quad 0 \leq x \leq L, \quad (5.42)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \quad (5.43)$$

Compatibility of the conditions (5.42) and (5.43) implies that $f(0) = f(L) = 0$. We seek the solution in the separated form

$$u(x, t) = P(x)Q(t). \quad (5.44)$$

Substituting (5.44) into (5.41) yields

$$\frac{Q_t}{kQ} = \frac{P_{xx}}{P}. \quad (5.45)$$

Since x and t are independent variables, it follows from (5.45) that both sides of the equation must be constant. Therefore

$$\frac{Q_t}{kQ} = \frac{P_{xx}}{P} = -\lambda \quad (5.46)$$

for some $\lambda \in \mathbb{R}$. The constant λ is called the *separation constant*, and the negative sign is chosen by convention. Thus the functions P and Q satisfy the ordinary differential equations

$$P_{xx} + \lambda P = 0, \quad 0 < x < L, \quad (5.47)$$

$$Q_t + k\lambda Q = 0, \quad t > 0. \quad (5.48)$$

The boundary conditions

$$u(0, t) = P(0)Q(t) = 0, \quad u(L, t) = P(L)Q(t) = 0 \quad (5.49)$$

imply that $P(0) = P(L) = 0$. The boundary value problem for the function P ,

$$P_{xx} + \lambda P = 0, \quad (5.50)$$

$$P(0) = P(L) = 0, \quad (5.51)$$

is called the *Sturm–Liouville problem* associated with the equation (5.41)–(5.43). If equation (5.50) with the boundary conditions (5.51) has a nontrivial solution $P \neq 0$ for some $\lambda \in \mathbb{R}$, then P is called an *eigenfunction* and λ an *eigenvalue* of the Sturm–Liouville problem.

The first step in solving the heat equation is therefore to determine the eigenfunctions and eigenvalues of this problem. Since the nature of the solutions depends on the sign of the constant λ , we will consider separately the cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. We introduce the notation $\lambda = \pm c^2$, $c \geq 0$.

Case $\lambda = -c^2 < 0$. The general solution of equation (5.50) is

$$P(x) = Ae^{cx} + Be^{-cx}. \quad (5.52)$$

The boundary conditions imply that A and B satisfy the system of equations

$$P(0) = A + B = 0, \quad (5.53)$$

$$P(L) = Ae^{cL} + Be^{-cL} = 0. \quad (5.54)$$

The determinant of the matrix of this system is

$$\begin{vmatrix} 1 & 1 \\ e^{cL} & e^{-cL} \end{vmatrix} \neq 0 \quad (5.55)$$

and therefore the system has only the trivial solution $A = B = 0$. Hence $\lambda < 0$ is not an eigenvalue of the Sturm–Liouville problem.

Case $\lambda = 0$. In this case we have

$$P(x) = A + Bx \quad (5.56)$$

which yields

$$P(0) = A = 0, \quad (5.57)$$

$$P(L) = A + BL = 0. \quad (5.58)$$

Clearly $A = B = 0$, and therefore $\lambda = 0$ is not an eigenvalue of the problem (5.50)–(5.51).

Case $\lambda = c^2 > 0$. The general solution of equation (5.50) is a linear combination

$$P(x) = A \cos(cx) + B \sin(cx). \quad (5.59)$$

From the boundary conditions we obtain

$$P(0) = A = 0, \quad (5.60)$$

$$P(L) = A \cos(cL) + B \sin(cL) = 0. \quad (5.61)$$

This system has a nontrivial solution $B \neq 0$ only if the constant c satisfies

$$\sin(cL) = 0. \quad (5.62)$$

This equation has the discrete solutions

$$c_n = \frac{n\pi}{L}, \quad n = \pm 1, \pm 2, \dots \quad (5.63)$$

and therefore the eigenvalues of the Sturm–Liouville problem are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \geq 1. \quad (5.64)$$

To each eigenvalue λ_n there corresponds the eigenfunction

$$P_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right), \quad n \geq 1. \quad (5.65)$$

We conclude that the Sturm–Liouville problem (5.50)–(5.51) has infinitely many solutions (5.65) with the corresponding eigenvalues (5.64).

For each eigenvalue λ_n the function Q satisfies the corresponding equation

$$Q_t + k\lambda_n Q = 0 \quad (5.66)$$

which has the exponential solution

$$Q_n(t) = e^{-k\left(\frac{n\pi}{L}\right)^2 t}, \quad n \geq 1. \quad (5.67)$$

By this method we obtain a sequence of separated solutions

$$u_n(x, t) = Q_n(t)P_n(x) = B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right), \quad n \geq 1 \quad (5.68)$$

which satisfy the boundary conditions $u_n(0, t) = u_n(L, t) = 0$. By the principle of superposition, every linear combination

$$u(x, t) = \sum_{n=1}^N u_n(x, t) = \sum_{n=1}^N B_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) \quad (5.69)$$

is also a solution of the heat equation satisfying $u(0, t) = u(L, t) = 0$.

At the initial time $t = 0$ we have $u(x, 0) = f(x)$. If the given function f can be written as the linear combination

$$f(x) = \sum_{n=1}^N B_n \sin\left(\frac{n\pi}{L}x\right), \quad 0 \leq x \leq L, \quad (5.70)$$

then the function (5.69) is a solution of our problem (5.41)–(5.43).

Example 5.1 Determine the solution of the problem

$$u_t - 2u_{xx} = 0, \quad 0 < x < \pi, \quad t > 0, \quad (5.71)$$

$$u(x, 0) = 5 \sin(2x) - 10 \sin(3x), \quad 0 \leq x \leq \pi, \quad (5.72)$$

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0. \quad (5.73)$$

Solution. In this example $L = \pi$ and $k = 2$, therefore the solution has the form

$$u(x, t) = \sum_{n=1}^N B_n e^{-2n^2 t} \sin(nx). \quad (5.74)$$

From the initial condition

$$u(x, 0) = \sum_{n=1}^N B_n \sin(nx) = 5 \sin(2x) - 10 \sin(3x) \quad (5.75)$$

we conclude that $N = 3$, $B_1 = 0$, $B_2 = 5$, and $B_3 = -10$. Substituting these values into equation (5.74) we obtain

$$u(x, t) = 5e^{-8t} \sin(2x) - 10e^{-18t} \sin(3x). \quad (5.76)$$

□

It is clear that an arbitrary function f cannot, in general, be written as a finite linear combination (5.70). However, if f can be expanded into a Fourier series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) \quad (5.77)$$

on the interval $[0, L]$, then we expect that the solution of the problem is given in the form of the series

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t).$$

The following theorem gives conditions which guarantee that the series $\sum_{n=1}^{\infty} u_n(x, t)$ is a classical solution of the heat equation.

Theorem 5.4 (Existence of a Solution) Suppose that the function $f: [0, L] \rightarrow \mathbb{R}$ satisfies

(i) f is continuous on $[0, L]$ and piecewise C^1 on $[0, L]$,

(ii) $f(0) = f(L) = 0$.

Then the function

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right), \quad B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \quad (5.78)$$

is a classical solution of the initial-boundary value problem

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (5.79)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L, \quad (5.80)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \quad (5.81)$$

Proof. Let \tilde{f} be the odd extension of the function f ,

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ -f(-x), & -L \leq x < 0. \end{cases} \quad (5.82)$$

The function \tilde{f} is continuous and piecewise C^1 on $[-L, L]$, and clearly $\tilde{f}(-L) = \tilde{f}(L) = 0$. By Theorem 2.4 the Fourier series

$$\tilde{f}(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \quad (5.83)$$

converges uniformly to \tilde{f} on $[-L, L]$. The Fourier coefficients are

$$A_n = \frac{1}{L} \int_{-L}^L \tilde{f}(x) \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad n \geq 0, \quad (5.84)$$

$$B_n = \frac{1}{L} \int_{-L}^L \tilde{f}(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n \geq 1, \quad (5.85)$$

and $\sum_{n=1}^{\infty} |B_n| < \infty$. The functions

$$u_n(x, t) = B_n e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right), \quad n \geq 1 \quad (5.86)$$

satisfy the heat equation and the boundary conditions $u_n(0, t) = u_n(L, t) = 0$. Since the functions $u_n(x, t)$ are bounded by the convergent series

$$|u_n(x, t)| \leq |B_n| \quad \text{for all } 0 \leq x \leq L, t \geq 0, \quad (5.87)$$

the Weierstrass test implies that the series $\sum_{n=1}^{\infty} u_n(x, t)$ converges uniformly on the closed set

$$\bar{\Omega} = \{(x, t) \mid 0 \leq x \leq L, t \geq 0\}.$$

Since the functions $u_n(x, t)$ are continuous on $\bar{\Omega}$, the sum of the series is also continuous,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t), \quad (x, t) \in \bar{\Omega}. \quad (5.88)$$

We now show that the function $u(x, t)$ satisfies the heat equation on Ω . Let $\varepsilon > 0$ and define

$$\Omega_\varepsilon = \{(x, t) \mid 0 < x < L, t > \varepsilon\}.$$

Since f is bounded we have

$$|B_n| = \frac{2}{L} \left| \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \right| \leq \frac{2}{L} \int_0^L |f(x)| dx \leq 2M, \quad (5.89)$$

where $M = \max_{x \in [-L, L]} |f(x)|$. Differentiating with respect to t we obtain

$$\frac{\partial u_n}{\partial t} = -B_n k \left(\frac{n\pi}{L}\right)^2 e^{-k\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right). \quad (5.90)$$

Thus at every point $(x, t) \in \Omega_\varepsilon$ we have the bound

$$\left| \frac{\partial u_n}{\partial t} \right| \leq |B_n| k \left(\frac{n\pi}{L}\right)^2 e^{-k\left(\frac{n\pi}{L}\right)^2 t} \leq 2Mk \left(\frac{\pi}{L}\right)^2 n^2 e^{-k\left(\frac{n\pi}{L}\right)^2 \varepsilon}. \quad (5.91)$$

It is easy to verify that the series

$$\sum_{n=1}^{\infty} n^2 e^{-k\left(\frac{n\pi}{L}\right)^2 \varepsilon}$$

converges. By the Weierstrass test it follows that the series $\sum_{n=1}^{\infty} \partial u_n / \partial t$ converges uniformly on Ω_ε . Therefore the function u can be differentiated term by term and we obtain

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{\partial u_n}{\partial t}, \quad (x, t) \in \Omega_\varepsilon. \quad (5.92)$$

Similarly one proves that

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}, \quad (x, t) \in \Omega_\varepsilon. \quad (5.93)$$

From (5.92) and (5.93) it follows that u satisfies

$$u_t - ku_{xx} = \frac{\partial}{\partial t} \left(\sum_{n=1}^{\infty} u_n \right) - k \frac{\partial^2}{\partial x^2} \left(\sum_{n=1}^{\infty} u_n \right) = \sum_{n=1}^{\infty} \left(\frac{\partial u_n}{\partial t} - k \frac{\partial^2 u_n}{\partial x^2} \right) = 0 \quad (5.94)$$

on Ω_ε . Since $\varepsilon > 0$ was arbitrary, we conclude that u is a solution of the heat equation on the open set

$$\Omega = \{(x, t) \mid 0 < x < L, t > 0\}.$$

The function u clearly satisfies the boundary conditions $u(0, t) = u(L, t) = 0$ and the initial condition

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin \left(\frac{n\pi}{L} x \right) = f(x), \quad 0 \leq x \leq L, \quad (5.95)$$

because the Fourier series (5.95) converges uniformly to f on $[0, L]$. ■

Neumann Boundary Conditions

In a similar way, the method of separation of variables can be used to derive the solution of the Neumann problem

$$u_t - ku_{xx} = 0, \quad 0 < x < L, t > 0, \quad (5.96)$$

$$u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0, \quad (5.97)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L. \quad (5.98)$$

The function f satisfies the compatibility conditions of the initial and boundary data $f'(0) = f'(L) = 0$. It can be shown that the solution is given by the series

$$u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-k(\frac{n\pi}{L})^2 t} \cos \left(\frac{n\pi}{L} x \right), \quad (5.99)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \left(\frac{n\pi}{L} x \right) dx, \quad n \geq 0. \quad (5.100)$$

Periodic Boundary Conditions

Suppose that a wire of length $2L$ is bent into a circle. At the points $x = -L$ and $x = L$ the temperature and its gradient take the same values, so the function $u(x, t)$

satisfies the equation

$$u_t - ku_{xx} = 0, \quad -L < x < L, \quad t > 0, \quad (5.101)$$

$$u(x, 0) = f(x), \quad -L \leq x \leq L, \quad (5.102)$$

with periodic boundary conditions

$$u(-L, t) = u(L, t), \quad u_x(-L, t) = u_x(L, t), \quad t \geq 0. \quad (5.103)$$

By separation of variables one can show that the general solution is

$$u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} e^{-k\left(\frac{n\pi}{L}\right)^2 t} \left[A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right) \right] \quad (5.104)$$

where

$$A_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad n \geq 0, \quad (5.105)$$

$$B_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx, \quad n \geq 1. \quad (5.106)$$

Example 5.2 Determine the solution of the problem

$$u_t - ku_{xx} = 0, \quad -L < x < L, \quad t > 0, \quad (5.107)$$

$$u(x, 0) = \cos^3\left(\frac{\pi}{L}x\right), \quad -L \leq x \leq L, \quad (5.108)$$

$$u(-L, t) = u(L, t), \quad u_x(-L, t) = u_x(L, t), \quad t \geq 0. \quad (5.109)$$

Solution. The function $f(x) = \cos^3\left(\frac{\pi}{L}x\right)$ can be expanded into a Fourier series on $[-L, L]$ using the trigonometric identity

$$\cos(x) \cos(y) = \frac{1}{2} [\cos(x+y) + \cos(x-y)]. \quad (5.110)$$

From (5.110) we obtain

$$\begin{aligned} \cos^3(x) &= \cos(x) \cos^2(x) = \frac{1}{2} \cos(x) [\cos(2x) + 1] \\ &= \frac{1}{2} \cos(x) \cos(2x) + \frac{1}{2} \cos(x) \\ &= \frac{1}{4} [\cos(3x) + \cos(x)] + \frac{1}{2} \cos(x) \\ &= \frac{3}{4} \cos(x) + \frac{1}{4} \cos(3x). \end{aligned} \quad (5.111)$$

Thus

$$\cos^3\left(\frac{\pi}{L}x\right) = \frac{3}{4}\cos\left(\frac{\pi}{L}x\right) + \frac{1}{4}\cos\left(\frac{3\pi}{L}x\right). \quad (5.112)$$

Equation (5.112) represents the Fourier expansion of the function $\cos^3(\pi x/L)$

$$\cos^3\left(\frac{\pi}{L}x\right) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{n\pi}{L}x\right) \right] \quad (5.113)$$

where $A_1 = \frac{3}{4}$ and $A_3 = \frac{1}{4}$, while all other coefficients A_n and B_n vanish. Substituting these coefficients into (5.104) we obtain

$$u(x, t) = \frac{3}{4}e^{-k(\frac{\pi}{L})^2t} \cos\left(\frac{\pi}{L}x\right) + \frac{1}{4}e^{-k(\frac{3\pi}{L})^2t} \cos\left(\frac{3\pi}{L}x\right). \quad (5.114)$$

□

Other Boundary Conditions

Finally, we note that the method of separation of variables can also be used to construct solutions of the heat equation for various combinations of boundary conditions. For example, one end of the interval may have a Dirichlet condition while the other has a Neumann condition,

$$u_t - ku_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (5.115)$$

$$u(0, t) = 0, \quad u_x(L, t) = 0, \quad t \geq 0, \quad (5.116)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L. \quad (5.117)$$

In this case the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-k(\frac{\pi}{2L})^2(2n-1)^2t} \sin\left(\frac{(2n-1)\pi}{2L}x\right) \quad (5.118)$$

where

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{(2n-1)\pi}{2L}x\right) dx. \quad (5.119)$$

The coefficients C_n can be determined by extending the function f from the interval $[0, L]$ to $[0, 2L]$. The extended function

$$\tilde{f}(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ f(2L - x), & L \leq x \leq 2L \end{cases} \quad (5.120)$$

is expanded into a Fourier series in the functions $\sin(n\pi x/(2L))$ on the interval $[0, 2L]$ (using an odd extension to the interval $[-2L, 2L]$), from which the coefficients C_n are obtained.

5.3 Separation of Variables for the Nonhomogeneous Equation

By a modification of the previous method it is possible to determine the solution of the nonhomogeneous heat equation

$$u_t - ku_{xx} = F(x, t), \quad 0 < x < L, \quad t > 0, \quad (5.121)$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L, \quad (5.122)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \quad (5.123)$$

If $F = 0$, then we know that the solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k(\frac{n\pi}{L})^2 t} \sin\left(\frac{n\pi}{L}x\right), \quad (5.124)$$

which may be interpreted as a Fourier series whose coefficients

$$B_n(t) = B_n e^{-k(\frac{n\pi}{L})^2 t}$$

depend on the parameter t . This suggests that we seek the solution of the nonhomogeneous equation by the method of variation of parameters in the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right), \quad (5.125)$$

where $T_n(t)$ are unknown functions. The functions $T_n(t)$ can be determined if $F(x, t)$ can be expanded into a Fourier series of the same form as (5.125).

Assume that $F(x, t)$ is continuous and piecewise C^1 in the variable $x \in [0, L]$ for each $t \geq 0$. Since the function $F(x, t)$ models an internal source which heats or cools the rod, it is reasonable to assume that $F(0, t) = F(L, t) = 0$. Then the Fourier series

$$F(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi}{L}x\right) \quad (5.126)$$

converges uniformly to $F(x, t)$ on the interval $[0, L]$ for each $t \geq 0$. Substituting (5.125) and (5.126) into the heat equation, we obtain

$$\sum_{n=1}^{\infty} \left(T_n'(t) + k \left(\frac{n\pi}{L} \right)^2 T_n(t) \right) \sin \left(\frac{n\pi}{L} x \right) = \sum_{n=1}^{\infty} F_n(t) \sin \left(\frac{n\pi}{L} x \right). \quad (5.127)$$

It follows that the functions $T_n(t)$ satisfy the differential equations

$$T_n'(t) + k \left(\frac{n\pi}{L} \right)^2 T_n(t) = F_n(t), \quad n \geq 1. \quad (5.128)$$

The general solution of equation (5.128) is

$$T_n(t) = B_n e^{-k \left(\frac{n\pi}{L} \right)^2 t} + T_n^p(t), \quad (5.129)$$

where B_n is a constant of integration and $T_n^p(t)$ is a particular solution depending on the function $F_n(t)$. Substituting the solution (5.129) into (5.78), we obtain

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L} \right)^2 t} \sin \left(\frac{n\pi}{L} x \right) + \sum_{n=1}^{\infty} T_n^p(t) \sin \left(\frac{n\pi}{L} x \right). \quad (5.130)$$

We recognize the first term in (5.130) as the solution of the homogeneous problem,

$$u_h(x, t) = \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L} \right)^2 t} \sin \left(\frac{n\pi}{L} x \right), \quad (5.131)$$

while the second term is a particular solution,

$$u_p(x, t) = \sum_{n=1}^{\infty} T_n^p(t) \sin \left(\frac{n\pi}{L} x \right), \quad (5.132)$$

which depends on the function $F(x, t)$. The coefficients B_n are determined from the initial condition $u(x, 0) = f(x)$,

$$\sum_{n=1}^{\infty} (B_n + T_n^p(0)) \sin \left(\frac{n\pi}{L} x \right) = f(x), \quad 0 \leq x \leq L, \quad (5.133)$$

from which we obtain

$$B_n + T_n^p(0) = \frac{2}{L} \int_0^L f(x) \sin \left(\frac{n\pi}{L} x \right) dx. \quad (5.134)$$

Thus the solution of the nonhomogeneous heat equation is completely determined.

Example 5.3 Solve the nonhomogeneous equation

$$u_t - u_{xx} = e^{-t} \sin(3x), \quad 0 < x < \pi, \quad t > 0, \quad (5.135)$$

$$u(0, t) = u(\pi, t) = 0, \quad t \geq 0, \quad (5.136)$$

$$u(x, 0) = x \sin(x), \quad 0 \leq x \leq \pi. \quad (5.137)$$

Solution. The nonhomogeneous term $F(x, t) = e^{-t} \sin(3x)$ satisfies the condition $F(0, t) = F(\pi, t) = 0$, so we seek the solution in the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin(nx). \quad (5.138)$$

Substituting (5.138) into equation (5.135), we obtain

$$\sum_{n=1}^{\infty} (T'_n(t) + n^2 T_n(t)) \sin(nx) = e^{-t} \sin(3x). \quad (5.139)$$

It follows that the functions $T_n(t)$ satisfy the differential equations

$$T'_n(t) + n^2 T_n(t) = 0, \quad n \neq 3, \quad (5.140)$$

$$T'_3(t) + 9T_3(t) = e^{-t}. \quad (5.141)$$

The solution of the first equation is

$$T_n(t) = B_n e^{-n^2 t}, \quad n \neq 3. \quad (5.142)$$

The second equation can be solved by the method of variation of constants, setting $T_3(t) = C_3(t)e^{-9t}$. Then from (5.141) it follows that $C'_3(t) = e^{8t}$, which implies $C_3(t) = \frac{1}{8}e^{8t} + B_3$. Hence

$$T_3(t) = B_3 e^{-9t} + \frac{1}{8} e^{-t}. \quad (5.143)$$

Therefore $u(x, t)$ can be written as

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-n^2 t} \sin(nx) + \frac{1}{8} e^{-t} \sin(3x). \quad (5.144)$$

From the initial condition (5.137) we obtain

$$\sum_{n=1}^{\infty} B_n \sin(nx) + \frac{1}{8} \sin(3x) = x \sin(x), \quad 0 \leq x \leq \pi. \quad (5.145)$$

Define the coefficients $\tilde{B}_n = B_n$ for $n \neq 3$ and $\tilde{B}_3 = B_3 + \frac{1}{8}$, so that

$$\sum_{n=1}^{\infty} \tilde{B}_n \sin(nx) = x \sin(x), \quad 0 \leq x \leq \pi. \quad (5.146)$$

The coefficients \tilde{B}_n are the Fourier coefficients

$$\tilde{B}_n = \frac{2}{\pi} \int_0^{\pi} x \sin(x) \sin(nx) dx, \quad n \geq 1. \quad (5.147)$$

For $n = 1$ we have

$$\tilde{B}_1 = \frac{2}{\pi} \int_0^{\pi} x \sin^2(x) dx = \frac{\pi}{2}. \quad (5.148)$$

For $n \geq 2$ we obtain

$$\begin{aligned} \tilde{B}_n &= \frac{2}{\pi} \int_0^{\pi} x \sin(x) \sin(nx) dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \frac{1}{2} [\cos((n-1)x) - \cos((n+1)x)] dx \\ &= \frac{1}{\pi} \frac{1}{(n-1)^2} \left[\cos((n-1)x) + (n-1)x \sin(x) \sin((n-1)x) \right]_0^{\pi} \\ &\quad - \frac{1}{\pi} \frac{1}{(n+1)^2} \left[\cos((n+1)x) + (n+1)x \sin(x) \sin((n+1)x) \right]_0^{\pi} \\ &= \frac{1}{\pi} \frac{(-1)^{n-1} - 1}{(n-1)^2} - \frac{1}{\pi} \frac{(-1)^{n+1} - 1}{(n+1)^2} = -\frac{4n((-1)^n + 1)}{\pi(n^2 - 1)^2}. \end{aligned} \quad (5.149)$$

Note that $\tilde{B}_n = 0$ for odd $n > 1$, while for even indices we have

$$\tilde{B}_{2n} = -\frac{16}{\pi} \frac{n}{(4n^2 - 1)^2}, \quad n = 1, 2, 3, \dots \quad (5.150)$$

From equations (5.148) and (5.150) we obtain

$$B_1 = \frac{\pi}{2}, \quad B_3 = -\frac{1}{8}, \quad B_n = 0 \quad \text{for odd } n > 3, \quad (5.151)$$

$$B_{2n} = -\frac{16}{\pi} \frac{n}{(4n^2 - 1)^2}, \quad n = 1, 2, 3, \dots \quad (5.152)$$

Substituting the coefficients B_n into equation (5.144), we find the solution of the problem:

$$u(x, t) = \frac{\pi}{2} e^{-t} \sin(x) + \frac{1}{8} (e^{-t} - e^{-9t}) \sin(3x) - \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{n}{(4n^2 - 1)^2} e^{-4n^2 t} \sin(2nx). \quad (5.153)$$

The graph of the function u is shown in Figure 5.2. \square

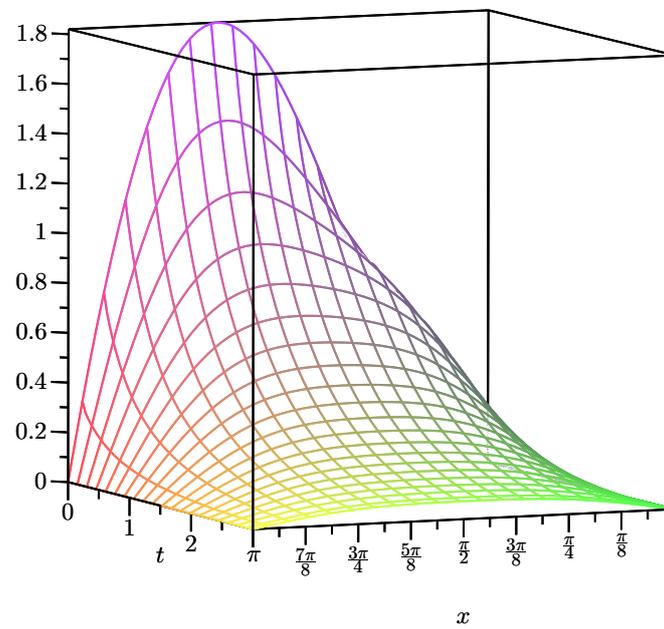


Figure 5.2: Graph of the function (5.153).

Chapter 6

The Wave Equation

6.1 Wave Motion and d'Alembert's Solution

The wave equation occupies a prominent place in applications because it describes oscillations of continuous mechanical media, the propagation of electromagnetic and sound waves, and also appears in the quantum description of elementary particles. In this chapter we consider the wave equation in one spatial dimension

$$u_{tt} - c^2 u_{xx} = 0. \quad (6.1)$$

Equation (6.1) describes the vibration of an elastic string in the idealized case in which dissipative effects such as internal friction of the string or air resistance may be neglected. The string lies along the x -axis, and the displacement $u(x, t)$ of the string from its equilibrium position is perpendicular to the x -axis. Such oscillation is called *transverse*, in contrast to *longitudinal* oscillation, which takes place along the x -axis. The constant $c > 0$ represents the speed of propagation of the wave.

We begin the study of the wave equation with some general observations about wave motion. Introduce the new variables

$$\alpha = x + ct, \quad \beta = x - ct \quad (6.2)$$

and the function $w(\alpha, \beta) = u(x(\alpha, \beta), t(\alpha, \beta))$. Then

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\alpha\beta}, \quad (6.3)$$

so in the new variables the equation takes the canonical form

$$w_{\alpha\beta} = 0. \quad (6.4)$$

Integrating this equation, we obtain

$$w(\alpha, \beta) = A(\alpha) + B(\beta), \quad (6.5)$$

and therefore the general solution is given by

$$u(x, t) = A(x + ct) + B(x - ct), \quad (6.6)$$

where A and B are arbitrary C^2 functions. The function $A(x + ct)$ represents a wave moving with speed c in the negative direction, while $B(x - ct)$ represents a wave moving with the same speed in the positive direction. Thus the general solution of the wave equation is a superposition of two waves travelling in opposite directions.

Suppose now that the wave equation describes the vibration of a very long string. Since wave motion propagates with finite speed, we may neglect the boundary conditions at the ends of the string, at least over some time interval. In this case the vibration of the string may be modelled by the equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (6.7)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}, \quad (6.8)$$

where the functions $f(x)$ and $g(x)$ represent the initial displacement and the initial velocity at the point x , respectively. Substituting the general solution into the initial conditions (6.8), we obtain

$$u(x, 0) = A(x) + B(x) = f(x), \quad (6.9)$$

$$u_t(x, 0) = cA'(x) - cB'(x) = g(x). \quad (6.10)$$

Integrating equation (6.10), it follows that

$$A(x) - B(x) = \frac{1}{c} \int_0^x g(s) ds + D, \quad (6.11)$$

where D is a constant of integration. Hence the functions A and B satisfy the system of equations

$$A(x) + B(x) = f(x), \quad (6.12)$$

$$A(x) - B(x) = \frac{1}{c} \int_0^x g(s) ds + D, \quad (6.13)$$

which has the unique solution

$$A(x) = \frac{1}{2}f(x) + \frac{1}{2c} \int_0^x g(s) ds + \frac{D}{2}, \quad (6.14)$$

$$B(x) = \frac{1}{2}f(x) - \frac{1}{2c} \int_0^x g(s) ds - \frac{D}{2}. \quad (6.15)$$

Therefore the solution of the wave equation is given by

$$u(x, t) = A(x + ct) + B(x - ct) \quad (6.16)$$

$$= \frac{1}{2} \left[f(x + ct) + f(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds. \quad (6.17)$$

This solution is called the *d'Alembert solution*, with initial displacement $f \in C^2(\mathbb{R})$ and initial velocity $g \in C(\mathbb{R})$. Intuitively, it is clear that the motion of the string is uniquely determined if the initial position and velocity are known. It is also reasonable to expect that small changes in the initial conditions cause only small changes in the solution $u(x, t)$ when the motion is observed over a finite time interval. We prove these claims in the following theorem.

Theorem 6.1 *Let $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$. Then the wave equation (6.7)–(6.8) has the unique solution*

$$u(x, t) = \frac{1}{2} \left[f(x + ct) + f(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds \quad (6.18)$$

which is stable with respect to the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$, $x \in \mathbb{R}$, on every finite interval $0 \leq t \leq T$.

Proof. Using the formula

$$\frac{d}{dx} \int_{\varphi_1(x)}^{\varphi_2(x)} f(u) du = f(\varphi_2(x)) \varphi_2'(x) - f(\varphi_1(x)) \varphi_1'(x), \quad (6.19)$$

it is easy to verify that the function (6.18) satisfies equation (6.7) and the initial conditions (6.8). From the construction of the solution it follows that every function satisfying (6.7)–(6.8) must necessarily be of the form (6.18), and therefore the solution is unique.

Let us show that the solution (6.18) is stable. Let u_1 and u_2 be the solutions corresponding to the initial data f_1, g_1 and f_2, g_2 , respectively. Consider u_1 and u_2 on the time interval $[0, T]$. Let $\varepsilon > 0$ and assume that the initial data satisfy

$$\sup_{x \in \mathbb{R}} |f_1(x) - f_2(x)| < \frac{\varepsilon}{1+T}, \quad \sup_{x \in \mathbb{R}} |g_1(x) - g_2(x)| < \frac{\varepsilon}{1+T}. \quad (6.20)$$

The difference of the two solutions is

$$\begin{aligned} u_1(x, t) - u_2(x, t) &= \frac{1}{2} [f_1(x+ct) - f_2(x+ct) + f_1(x-ct) - f_2(x-ct)] \\ &\quad + \frac{1}{2c} \int_{x-ct}^{x+ct} (g_1(s) - g_2(s)) ds, \end{aligned} \quad (6.21)$$

and so from inequality (6.20) it follows that

$$\begin{aligned} |u_1(x, t) - u_2(x, t)| &\leq \frac{1}{2} (|f_1(x+ct) - f_2(x+ct)| + |f_1(x-ct) - f_2(x-ct)|) \\ &\quad + \frac{1}{2c} \int_{x-ct}^{x+ct} |g_1(s) - g_2(s)| ds \\ &\leq \frac{1}{2} \left(\frac{\varepsilon}{1+T} + \frac{\varepsilon}{1+T} \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \frac{\varepsilon}{1+T} ds \\ &= \frac{\varepsilon}{1+T} (1+t) \leq \varepsilon \end{aligned} \quad (6.22)$$

for every $0 \leq t \leq T$. This implies that

$$\sup_{\substack{x \in \mathbb{R} \\ 0 \leq t \leq T}} |u_1(x, t) - u_2(x, t)| \leq \varepsilon. \quad (6.23)$$

We have thus shown that for every $\varepsilon > 0$ there exists $\delta = 2\varepsilon/(1+T)$ such that

$$\sup_{x \in \mathbb{R}} (|f_1(x) - f_2(x)| + |g_1(x) - g_2(x)|) < \delta \quad \Rightarrow \quad \sup_{\substack{x \in \mathbb{R} \\ 0 \leq t \leq T}} |u_1(x, t) - u_2(x, t)| < \varepsilon. \quad (6.24)$$

Therefore, on every finite interval $0 \leq t \leq T$, a small change in the initial conditions causes only a small change in the solution, and hence the solution of the wave equation is stable. ■

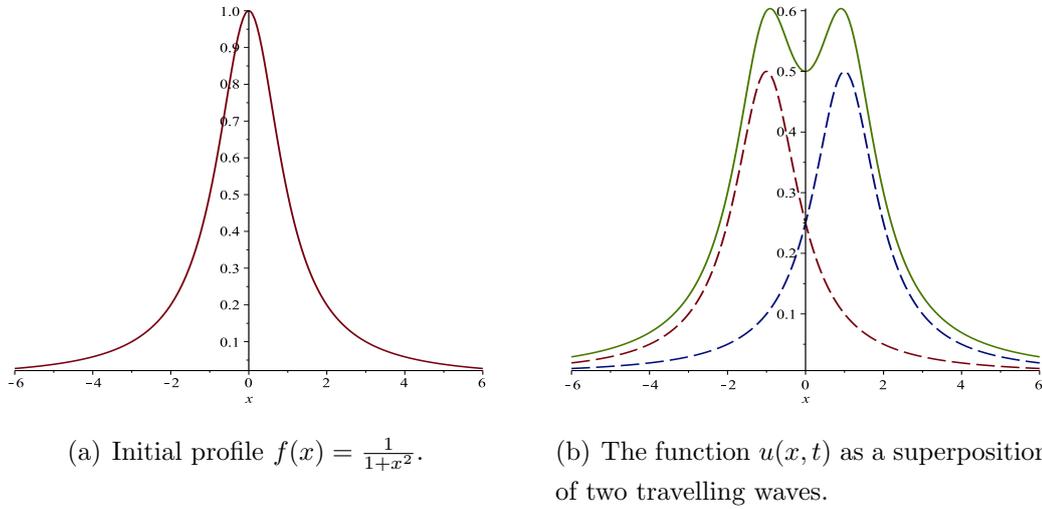


Figure 6.1:

Example 6.1 Solve the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (6.25)$$

$$u(x, 0) = \frac{1}{1+x^2}, \quad u_t(x, 0) = 0. \quad (6.26)$$

Solution. In this problem we have $f(x) = \frac{1}{1+x^2}$ and $g(x) = 0$. The d'Alembert solution is given by

$$u(x, t) = \frac{1}{2} \left[\frac{1}{1+(x+ct)^2} + \frac{1}{1+(x-ct)^2} \right]. \quad (6.27)$$

Figure 6.1 shows the initial wave profile and the solution $u(x, t)$. \square

Example 6.2 Determine the solution of the problem

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (6.28)$$

$$u(x, 0) = \sin(x), \quad u_t(x, 0) = \cos(x). \quad (6.29)$$

Solution. By d'Alembert's formula we have

$$\begin{aligned} u(x, t) &= \frac{1}{2} [\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \cos(s) ds \\ &= \sin(x) \cos(ct) + \frac{1}{2c} [\sin(x+ct) - \sin(x-ct)] \\ &= \sin(x) \cos(ct) + \frac{1}{c} \cos(x) \sin(ct). \end{aligned} \quad (6.30)$$

Using the identity

$$\sin(\alpha) \cos(\beta) = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)],$$

the solution can be written as a superposition of waves travelling in opposite directions:

$$u(x, t) = \frac{c+1}{2c} \sin(x+ct) + \frac{c-1}{2c} \sin(x-ct). \quad (6.31)$$

□

6.2 D'Alembert's Solution for the Nonhomogeneous Wave Equation

In this section we extend d'Alembert's solution to the nonhomogeneous wave equation

$$u_{tt} - c^2 u_{xx} = F(x, t), \quad x \in \mathbb{R}, \quad t > 0, \quad (6.32)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}. \quad (6.33)$$

The function $F(x, t)$ models an external force acting on the string at the point x and at time t . Introduce the variable $y = ct$ and define the function $w(x, y) = u(x, t)$.

With this transformation we obtain the equivalent problem

$$w_{xx} - w_{yy} = F^*(x, y), \quad x \in \mathbb{R}, \quad y > 0, \quad (6.34)$$

$$w(x, 0) = f(x), \quad w_y(x, 0) = g^*(x), \quad x \in \mathbb{R} \quad (6.35)$$

where

$$F^*(x, y) = -\frac{1}{c^2} F(x, t), \quad g^*(x) = \frac{1}{c} g(x). \quad (6.36)$$

Let (x_0, y_0) be an arbitrary point in the half-plane $x \in \mathbb{R}, y > 0$. Consider the triangle D with vertices $P_0 = (x_0, y_0)$, $P_1 = (x_0 - y_0, 0)$, and $P_2 = (x_0 + y_0, 0)$. Denote the sides of the triangle by B_0 , B_1 , and B_2 as in Figure 6.2. The idea behind the solution of (6.34)–(6.35) is as follows. We integrate equation (6.34) over the triangle D and then, by Green's theorem, replace the double integral by a line integral over the sides of the triangle. By computing these line integrals, we obtain the value of the function w at the point (x_0, y_0) . This determines the solution, since the point (x_0, y_0) was chosen arbitrarily.

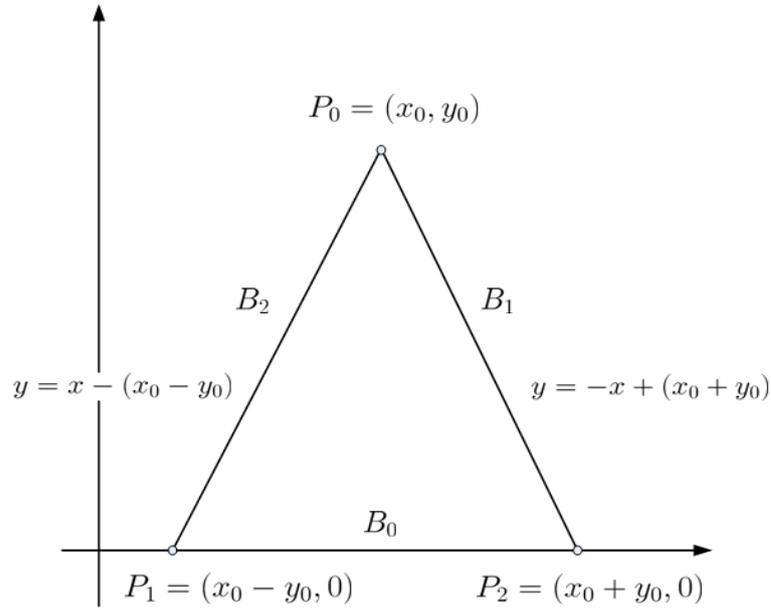


Figure 6.2: Integration region for the nonhomogeneous wave equation.

Integrating equation (6.34) over the triangle D , we obtain

$$\iint_D (w_{xx} - w_{yy}) dx dy = \iint_D F^*(x, y) dx dy. \quad (6.37)$$

By Green's theorem, the integral on the left-hand side equals

$$\iint_D (w_{xx} - w_{yy}) dx dy = \int_{\partial D} (w_y dx + w_x dy), \quad (6.38)$$

where ∂D is the positively oriented boundary of the triangle D , consisting of the segments B_0 , B_1 , and B_2 .

The integral over the segment B_0 is

$$\int_{B_0} (w_y dx + w_x dy) = \int_{x_0 - y_0}^{x_0 + y_0} w_y(x, 0) dx = \int_{x_0 - y_0}^{x_0 + y_0} g^*(x) dx. \quad (6.39)$$

The integral over the segment B_1 can be computed by introducing the parametrization

$$x(t) = x_0 + (1 - t)y_0, \quad y(t) = ty_0, \quad t \in [0, 1], \quad (6.40)$$

where we have taken the orientation of the segment into account. Then

$$\int_{B_1} (w_y dx + w_x dy) = \int_0^1 [w_y(x(t), y(t)) x'(t) + w_x(x(t), y(t)) y'(t)] dt. \quad (6.41)$$

Note that $x'(t) = -y'(t) = -y_0$, and therefore by replacing $x'(t)$ with $-y'(t)$ in the integral (6.41), we obtain

$$\begin{aligned} \int_{B_1} (w_y dx + w_x dy) &= - \int_0^1 [w_x(x(t), y(t)) x'(t) + w_y(x(t), y(t)) y'(t)] dt \\ &= - \int_0^1 \frac{d}{dt} w(x(t), y(t)) dt = -w(x(t), y(t)) \Big|_{t=0}^{t=1} \\ &= w(x_0 + y_0, 0) - w(x_0, y_0). \end{aligned} \quad (6.42)$$

Similarly, one shows that the integral over the segment B_2 is

$$\int_{B_2} (w_y dx + w_x dy) = w(x_0 - y_0, 0) - w(x_0, y_0). \quad (6.43)$$

Since $w(x, 0) = f(x)$, by summing the integrals over the segments B_0 , B_1 , and B_2 , we obtain

$$\int_{\partial D} (w_y dx + w_x dy) = f(x_0 + y_0) + f(x_0 - y_0) - 2w(x_0, y_0) + \int_{x_0 - y_0}^{x_0 + y_0} g^*(x) dx. \quad (6.44)$$

From equations (6.37), (6.38), and (6.44) it follows that

$$\begin{aligned} \iint_D F^*(x, y) dx dy &= \int_{\partial D} (w_y dx + w_x dy) \\ &= f(x_0 + y_0) + f(x_0 - y_0) - 2w(x_0, y_0) + \int_{x_0 - y_0}^{x_0 + y_0} g^*(x) dx. \end{aligned} \quad (6.45)$$

Hence we find

$$w(x_0, y_0) = \frac{1}{2} [f(x_0 + y_0) + f(x_0 - y_0)] + \frac{1}{2} \int_{x_0 - y_0}^{x_0 + y_0} g^*(x) dx - \frac{1}{2} \iint_D F^*(x, y) dx dy.$$

The function w at the point (x_0, y_0) is completely determined by the functions f , g , and F . Since the point $(x_0, y_0) \in \mathbb{R}_+^2$ was chosen arbitrarily, we may write

$$w(x, y) = \frac{1}{2} [f(x+y) + f(x-y)] + \frac{1}{2} \int_{x-y}^{x+y} g^*(x') dx' - \frac{1}{2} \iint_D F^*(x', y') dx' dy' \quad (6.46)$$

where it is understood that the vertex of the triangle D is at the point (x, y) . The original solution is then obtained from $u(x, t) = w(x, ct)$. Note that the first two terms in the above formula are solutions of the homogeneous wave equation, while the third term is a particular solution depending on the function $F(x, t)$.

Example 6.3 Determine the solution of the problem

$$w_{xx} - w_{yy} = 1, \quad x \in \mathbb{R}, \quad y > 0, \quad (6.47)$$

$$w(x, 0) = \sin(x), \quad w_y(x, 0) = x. \quad (6.48)$$

Solution. In this problem we have $F^*(x, y) = 1$, $f(x) = \sin(x)$, and $g^*(x) = x$. From d'Alembert's formula it follows that

$$w(x, y) = \frac{1}{2} [\sin(x+y) + \sin(x-y)] + \frac{1}{2} \int_{x-y}^{x+y} x' dx' - \frac{1}{2} \iint_D dx' dy' \quad (6.49)$$

$$= \sin(x) \cos(y) + \frac{1}{4} (x')^2 \Big|_{x-y}^{x+y} - \frac{1}{4} (2y)y \quad (6.50)$$

$$= \sin(x) \cos(y) + xy - \frac{1}{2} y^2. \quad (6.51)$$

□

Example 6.4 Solve the equation

$$u_{tt} - c^2 u_{xx} = x e^t, \quad x \in \mathbb{R}, \quad t > 0, \quad (6.52)$$

$$u(x, 0) = \sin(x), \quad u_t(x, 0) = 0. \quad (6.53)$$

Solution. Define the functions $F(x, t) = x e^t$, $f(x) = \sin(x)$, and $g(x) = 0$. Then

$$F^*(x, y) = -\frac{1}{c^2} x e^{y/c} \quad \text{and} \quad g^*(x) = 0 \quad (6.54)$$

and therefore, by d'Alembert's formula,

$$w(x, y) = \frac{1}{2} [\sin(x+y) + \sin(x-y)] + \frac{1}{2c^2} \iint_D x' e^{y'/c} dx' dy'. \quad (6.55)$$

Let the vertex of the triangle D be (x, y) (see Figure 6.2). Then

$$\begin{aligned} \iint_D x' e^{y'/c} dx' dy' &= \int_0^y \int_{y'+(x-y)}^{-y'+(x+y)} x' e^{y'/c} dx' dy' \\ &= \int_0^y e^{y'/c} \left(\frac{1}{2} (x')^2 \Big|_{y'+(x-y)}^{-y'+(x+y)} \right) dy' \\ &= \frac{1}{2} \int_0^y e^{y'/c} [(-y' + x + y)^2 - (y' + x - y)^2] dy' \\ &= 2x \int_0^y e^{y'/c} (-y' + y) dy' = 2c^2 x \left(e^{y/c} - \frac{y}{c} - 1 \right). \end{aligned} \quad (6.56)$$

Substituting (6.56) into (6.55), we obtain

$$w(x, y) = \sin(x) \cos(y) + x \left(e^{y/c} - \frac{y}{c} - 1 \right). \quad (6.57)$$

Therefore the solution $u(x, t)$ is given by

$$u(x, t) = w(x, ct) = \sin(x) \cos(ct) + x (e^t - t - 1). \quad (6.58)$$

□

6.3 Initial–Boundary Value Problem for the Wave Equation

Consider now the vibration of a string of length L under the action of an external force $F(x, t)$. This problem is described by the wave equation

$$u_{tt} - c^2 u_{xx} = F(x, t), \quad 0 < x < L, \quad t > 0, \quad (6.59)$$

together with the initial conditions

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L. \quad (6.60)$$

If the ends of the string are fixed, then u satisfies the Dirichlet conditions

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \quad (6.61)$$

If the ends of the string are free to oscillate in the direction perpendicular to the x -axis, then the string is positioned so that its tangent is horizontal at the points $x = 0$ and $x = L$. In that case the function u satisfies the Neumann conditions

$$u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0. \quad (6.62)$$

Clearly, if the string is fixed at one end and free at the other, then u satisfies mixed Dirichlet and Neumann conditions. We consider the solution of the wave equation on the domain

$$\Omega = \{(x, t) \mid 0 < x < L, t > 0\}.$$

Theorem 6.2 (Uniqueness of the Solution) *Let u_1 and u_2 be C^2 solutions of the problem (6.59)–(6.61). Then $u_1 = u_2$.*

Proof. Let $w = u_1 - u_2$. Then w is a solution of the homogeneous equation

$$w_{tt} - c^2 w_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (6.63)$$

$$w(x, 0) = w_t(x, 0) = 0, \quad 0 \leq x \leq L, \quad (6.64)$$

$$w(0, t) = w(L, t) = 0, \quad t \geq 0. \quad (6.65)$$

We will show that the problem (6.63)–(6.65) has only the trivial solution $w = 0$. Define the auxiliary function

$$E(t) = \frac{1}{2} \int_0^L (c^2 w_x^2 + w_t^2) dx. \quad (6.66)$$

The function $E(t)$ represents the total energy of the vibrating string at time t . The functions $(w_x^2)_t = 2w_x w_{xt}$ and $(w_t^2)_t = 2w_t w_{tt}$ are continuous, so by Leibniz's rule for differentiation under the integral sign we obtain

$$\frac{dE}{dt} = \int_0^L (c^2 w_x w_{xt} + w_t w_{tt}) dx. \quad (6.67)$$

Integrating the first term by parts, we find

$$\begin{aligned} \int_0^L w_x w_{xt} dx &= w_x w_t \Big|_{x=0}^{x=L} - \int_0^L w_t w_{xx} dx \\ &= w_x(L, t) w_t(L, t) - w_x(0, t) w_t(0, t) - \int_0^L w_t w_{xx} dx. \end{aligned} \quad (6.68)$$

The function w satisfies the boundary condition $w(0, t) = 0$ for every $t \geq 0$, which implies

$$w_t(0, t) = \lim_{\Delta t \rightarrow 0} \frac{w(0, t + \Delta t) - w(0, t)}{\Delta t} = 0. \quad (6.69)$$

Similarly, from the condition $w(L, t) = 0$ for every $t \geq 0$, it follows that

$$w_t(L, t) = 0. \quad (6.70)$$

Therefore

$$\int_0^L w_x w_{xt} dx = - \int_0^L w_t w_{xx} dx \quad (6.71)$$

and hence the derivative of the energy is

$$\frac{dE}{dt} = \int_0^L w_t (w_{tt} - c^2 w_{xx}) dx = 0 \quad (6.72)$$

since $w_{tt} - c^2 w_{xx} = 0$. We conclude that the function $E(t)$ is constant, that is,

$$E(t) = E(0) \quad \text{for every } t \geq 0. \quad (6.73)$$

At the initial time we have

$$E(0) = \frac{1}{2} \int_0^L [c^2 w_x^2(x, 0) + w_t^2(x, 0)] dx = \frac{1}{2} \int_0^L c^2 w_x^2(x, 0) dx \quad (6.74)$$

because of the initial condition $w_t(x, 0) = 0$. Since $w(x, 0) = 0$ for every $x \in [0, L]$, we have

$$w_x(x, 0) = \lim_{\Delta x \rightarrow 0} \frac{w(x + \Delta x, 0) - w(x, 0)}{\Delta x} = 0. \quad (6.75)$$

We conclude that $E(0) = 0$, which implies

$$E(t) = \frac{1}{2} \int_0^L (c^2 w_x^2 + w_t^2) dx = 0. \quad (6.76)$$

It follows that

$$w_x(x, t) = w_t(x, t) = 0, \quad (6.77)$$

so the function w is constant, since it does not depend on the variables x and t . Now from the initial condition $w(x, 0) = 0$ we conclude that $w(x, t) = 0$ for every $x \in [0, L]$ and $t \geq 0$. Therefore $u_1 = u_2$, which shows that the solution of the problem is unique. ■

Note that the same proof also implies uniqueness of the solution of the wave equation with Neumann boundary conditions $u_x(0, t) = 0$ and $u_x(L, t) = 0$.

6.3.1 Separation of Variables for the Homogeneous Equation

Dirichlet Boundary Conditions

Let us now consider the solution of the homogeneous wave equation with Dirichlet boundary conditions

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (6.78)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L, \quad (6.79)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \quad (6.80)$$

Compatibility of the initial and boundary conditions requires that f and g satisfy $f(0) = f(L) = 0$ and $g(0) = g(L) = 0$, since the initial displacement and velocity at the points $x = 0$ and $x = L$ are zero. We seek the solution of the problem in the separated form

$$u(x, t) = P(x)Q(t). \quad (6.81)$$

Substituting (6.81) into equation (6.78), we obtain $PQ_{tt} = c^2P_{xx}Q$, that is,

$$\frac{P_{xx}}{P} = \frac{1}{c^2} \frac{Q_{tt}}{Q}. \quad (6.82)$$

The variables x and t are independent, so both sides of equation (6.82) must be constant. Thus,

$$\frac{P_{xx}}{P} = \frac{1}{c^2} \frac{Q_{tt}}{Q} = -\lambda \quad (6.83)$$

for some $\lambda \in \mathbb{R}$, called the separation constant. It follows that the functions P and Q satisfy the ordinary differential equations

$$P_{xx} + \lambda P = 0, \quad 0 < x < L, \quad (6.84)$$

$$Q_{tt} + \lambda c^2 Q = 0, \quad t > 0. \quad (6.85)$$

The boundary conditions $u(0, t) = P(0)Q(t) = 0$ and $u(L, t) = P(L)Q(t) = 0$ imply $P(0) = 0$ and $P(L) = 0$. Therefore the function P satisfies the associated Sturm–Liouville problem

$$P_{xx} + \lambda P = 0, \quad (6.86)$$

$$P(0) = P(L) = 0. \quad (6.87)$$

In Section 5.2 it was shown that equation (6.86)–(6.87) has nontrivial solutions only for the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \geq 1, \quad (6.88)$$

with corresponding eigenfunctions

$$P_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right), \quad n \geq 1. \quad (6.89)$$

For each eigenvalue λ_n we can determine the function $Q(t)$ from equation (6.85). The general solution is

$$Q_n(t) = C_n \cos\left(\frac{n\pi c}{L}t\right) + D_n \sin\left(\frac{n\pi c}{L}t\right), \quad n \geq 1. \quad (6.90)$$

Thus we obtain the sequence of functions

$$u_n(x, t) = P_n(x)Q_n(t) = \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right), \quad (6.91)$$

where a_n and b_n are arbitrary constants. The functions $u_n(x, t)$ satisfy the wave equation (6.78) and the boundary conditions (6.80), and every linear combination

$$u(x, t) = \sum_{n=1}^N \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right) \quad (6.92)$$

is also a solution of the same problem.

The function $u_n(x, t)$ is called the *n*th-order harmonic for a string with fixed ends at $x = 0$ and $x = L$. If a_n and b_n are not both zero, then the harmonic $u_n(x, t)$ can be written in the following form. Define $R_n = \sqrt{a_n^2 + b_n^2}$. Since $(a_n/R_n)^2 + (b_n/R_n)^2 = 1$, there exists $\theta_n \in \mathbb{R}$ such that

$$a_n = R_n \sin(\theta_n) \quad \text{and} \quad b_n = R_n \cos(\theta_n). \quad (6.93)$$

Then

$$\begin{aligned} u_n(x, t) &= R_n \left[\sin(\theta_n) \cos\left(\frac{n\pi c}{L}t\right) + \cos(\theta_n) \sin\left(\frac{n\pi c}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right) \\ &= R_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi c}{L}t + \theta_n\right). \end{aligned} \quad (6.94)$$

At the point x , the harmonic $u_n(x, t)$ has amplitude $R_n \left| \sin\left(\frac{n\pi}{L}x\right) \right|$ and phase θ_n . The period of the harmonic, that is, the time required for one oscillation, is obtained from the equation $n\pi cT/L = 2\pi$, which gives

$$T_n = \frac{2L}{nc}. \quad (6.95)$$

The frequency of oscillation f_n is the number of oscillations per unit time,

$$f_n = \frac{1}{T_n} = \frac{nc}{2L}. \quad (6.96)$$

Figure 6.3 shows the first three harmonics for a vibrating string with fixed ends.

In general, the vibration of the string is a superposition of harmonics of all orders, that is, of all possible frequencies, so we expect that the general solution of the wave equation can be written in the form

$$u(x, t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right). \quad (6.97)$$

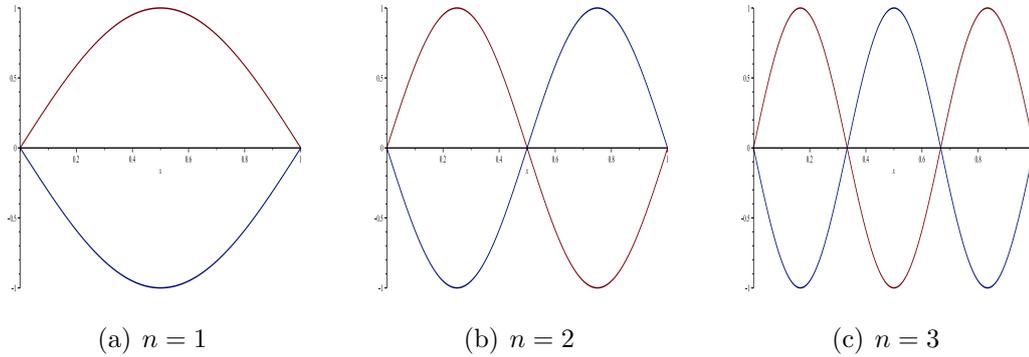


Figure 6.3: The first three harmonics for the vibration of a string with fixed ends.

The constants a_n and b_n are determined by the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. If we assume that the series (6.97) can be differentiated term by term, then we obtain

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x), \quad 0 \leq x \leq L, \quad (6.98)$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) b_n \sin\left(\frac{n\pi}{L}x\right) = g(x), \quad 0 \leq x \leq L. \quad (6.99)$$

By taking odd extensions of the functions f and g to $[-L, L]$, we can determine the Fourier coefficients

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx, \quad (6.100)$$

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx. \quad (6.101)$$

The resulting solution shows that the vibration of the string is completely determined by the initial and boundary conditions. This solution is still formal, because it remains to determine conditions under which the series (6.97) represents a C^2 function satisfying the wave equation.

Theorem 6.3 (Existence of a Solution) *Let $f \in C^4([0, L])$ and $g \in C^3([0, L])$. Assume that the functions f and g satisfy the conditions*

(i) $f(0) = f(L) = 0, f''(0) = f''(L) = 0,$

(ii) $g(0) = g(L) = 0$.

Then the function (6.97), where the Fourier coefficients are given by (6.100) and (6.101), is a classical solution of the wave equation with Dirichlet conditions

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \quad t > 0, \quad (6.102)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L, \quad (6.103)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0. \quad (6.104)$$

Proof. Let us show that the coefficients a_n and b_n tend to zero sufficiently rapidly so that the sum of the series (6.97) is a continuous function which can be differentiated term by term. Integrating relation (6.100) by parts, we obtain

$$a_n = \frac{2}{n\pi} \int_0^L f'(x) \cos\left(\frac{n\pi}{L}x\right) dx \quad (6.105)$$

since $f(0) = f(L) = 0$. Using the assumption $f''(0) = f''(L) = 0$, and iterating integration by parts three times, we find

$$a_n = \frac{2L^3}{(n\pi)^4} \int_0^L f^{(4)}(x) \sin\left(\frac{n\pi}{L}x\right) dx. \quad (6.106)$$

It follows that

$$\left|\left(\frac{n\pi}{L}\right)^2 a_n\right| \leq \frac{2L}{(n\pi)^2} \int_0^L |f^{(4)}(x)| dx. \quad (6.107)$$

Since $f^{(4)}$ is continuous on $[0, L]$, it is bounded on $[0, L]$, and we may define the constant

$$C_1 = \frac{2L}{\pi^2} \int_0^L |f^{(4)}(x)| dx. \quad (6.108)$$

Then

$$\left|\left(\frac{n\pi}{L}\right)^2 a_n\right| \leq \frac{C_1}{n^2}, \quad n \geq 1. \quad (6.109)$$

Similarly, using the conditions $g(0) = g(L) = 0$ and $g \in C^3([0, L])$, one shows from relation (6.101) that

$$\left|\left(\frac{n\pi}{L}\right)^2 b_n\right| \leq \frac{C_2}{n^2}, \quad n \geq 1, \quad (6.110)$$

where

$$C_2 = \frac{2L}{c\pi^2} \left(|g''(0)| + |g''(L)| + \int_0^L |g^{(3)}(x)| dx \right). \quad (6.111)$$

Define the functions

$$u_n(x, t) = \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right), \quad n \geq 1. \quad (6.112)$$

The inequalities (6.109) and (6.110) imply that the series $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ converge. Since

$$|u_n(x, t)| \leq |a_n| + |b_n| \quad (6.113)$$

for every $0 \leq x \leq L$ and $t \geq 0$, the Weierstrass test shows that the series $\sum_{n=1}^{\infty} u_n(x, t)$ converges uniformly on the set

$$\bar{\Omega} = \{(x, t) \mid 0 \leq x \leq L, t \geq 0\}$$

to a continuous function

$$u(x, t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right). \quad (6.114)$$

Let us show that the series (6.114) can be differentiated term by term on the open set $\Omega = \{(x, t) \mid 0 < x < L, t > 0\}$. Differentiating the function (6.114), we obtain

$$\frac{\partial^2 u_n}{\partial x^2} = -\left(\frac{n\pi}{L}\right)^2 \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \sin\left(\frac{n\pi}{L}x\right), \quad (6.115)$$

so inequalities (6.109) and (6.110) imply

$$\left| \frac{\partial^2 u_n}{\partial x^2} \right| \leq \left(\frac{n\pi}{L}\right)^2 |a_n| + \left(\frac{n\pi}{L}\right)^2 |b_n| \leq \frac{C_1 + C_2}{n^2} \quad (6.116)$$

for every $0 < x < L$ and $t > 0$. The series $\sum_{n=1}^{\infty} 1/n^2$ converges, and therefore by the Weierstrass test the series of functions $\sum_{n=1}^{\infty} \partial^2 u_n / \partial x^2$ converges uniformly on Ω . Hence the series $u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$ can be differentiated term by term, and

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2}, \quad (x, t) \in \Omega. \quad (6.117)$$

Differentiating with respect to t , we obtain

$$\left| \frac{\partial^2 u_n}{\partial t^2} \right| \leq c^2 \frac{C_1 + C_2}{n^2} \quad (6.118)$$

for every $0 < x < L$ and $t > 0$, so by the same argument we conclude that

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial t^2}, \quad (x, t) \in \Omega. \quad (6.119)$$

The functions u_n satisfy the wave equation $(u_n)_{tt} - c^2(u_n)_{xx} = 0$ and the boundary conditions $u_n(0, t) = u_n(L, t) = 0$. It follows that u satisfies

$$u_{tt} - c^2 u_{xx} = \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial t^2} - c^2 \sum_{n=1}^{\infty} \frac{\partial^2 u_n}{\partial x^2} = \sum_{n=1}^{\infty} \left(\frac{\partial^2 u_n}{\partial t^2} - c^2 \frac{\partial^2 u_n}{\partial x^2} \right) = 0 \quad (6.120)$$

together with the boundary conditions $u(0, t) = u(L, t) = 0$. Let us now consider the initial conditions

$$u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = f(x), \quad (6.121)$$

$$u_t(x, 0) = \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) b_n \sin\left(\frac{n\pi}{L}x\right) = g(x), \quad 0 \leq x \leq L. \quad (6.122)$$

If \tilde{f} and \tilde{g} are the odd extensions of the functions f and g to $[-L, L]$, then \tilde{f} and \tilde{g} are continuous and piecewise C^1 on $[-L, L]$, and satisfy $\tilde{f}(-L) = \tilde{f}(L) = 0$, $\tilde{g}(-L) = \tilde{g}(L) = 0$. By Theorem 2.4, the Fourier series (6.121) and (6.122) converge uniformly to f and g on the interval $[0, L]$, and the Fourier coefficients are given by

$$a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx, \quad (6.123)$$

$$\frac{n\pi c}{L} b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx, \quad (6.124)$$

that is,

$$b_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx. \quad (6.125)$$

Thus it has been shown that the function u satisfies the initial conditions (6.121) and (6.122). ■

In the proof of Theorem 6.3 we assumed that the functions f and g satisfy conditions which need not hold in applications. For example, if at the initial time the string is pulled transversely at a point $0 < x_0 < L$ and then released to vibrate freely, then the initial position is the triangular function

$$f(x) = \begin{cases} \frac{u_0}{x_0}x, & 0 \leq x \leq x_0, \\ u_0 \frac{x-L}{x_0-L}, & x_0 \leq x \leq L, \end{cases} \quad (6.126)$$

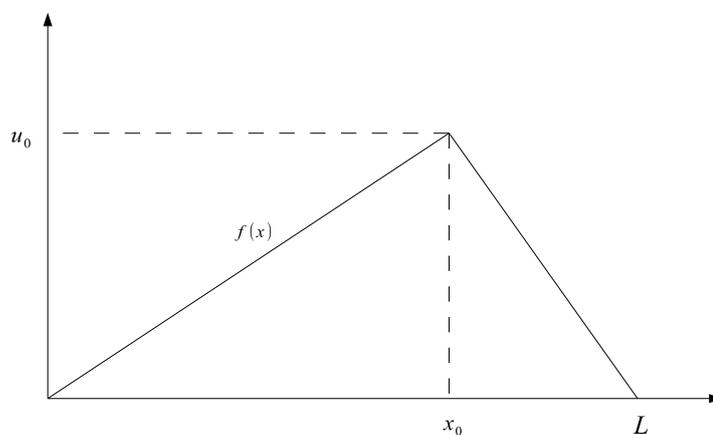


Figure 6.4: Graph of the function (6.126).

(see Figure 6.4). The function f is not differentiable at x_0 , so $f \notin C^1([0, L])$. To describe such initial conditions it is natural to assume only that the initial position is given by a continuous function $f \in C([0, L])$ such that $f(0) = f(L) = 0$. We would also like to allow the possibility that the initial velocity g has jump discontinuities of the first kind in order to describe the vibration of a string set in motion by the blow of a sharp object. The following example shows that under such weaker assumptions the solution of the wave equation is not a C^2 function, that is, it is not a classical solution in the sense of Definition 1.3. In this case it is necessary to extend the concept of solution of a differential equation to so-called weak solutions, which arise naturally in the formulation of partial differential equations via the calculus of variations. The study of weak solutions lies beyond the scope of these notes.

Example 6.5 (Motion of a Transversely Displaced String) *Determine the formal solution of the wave equation*

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \quad (6.127)$$

$$u(0, t) = u(L, t) = 0, \quad t \geq 0, \quad (6.128)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq L \quad (6.129)$$

where the function $f(x)$ is given by equation (6.126).

Solution. Since $u_t(x, 0) = 0$, we have $b_n = 0$ for every n . Therefore the formal solution has the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi c}{L}t\right) \sin\left(\frac{n\pi}{L}x\right) \quad (6.130)$$

where

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \\ &= \frac{2}{L} \int_0^{x_0} \frac{u_0}{x_0} x \sin\left(\frac{n\pi}{L}x\right) dx + \frac{2}{L} \int_{x_0}^L u_0 \frac{x-L}{x_0-L} \sin\left(\frac{n\pi}{L}x\right) dx. \end{aligned} \quad (6.131)$$

Integrating by parts, we obtain

$$\int_0^{x_0} x \sin\left(\frac{n\pi}{L}x\right) dx = \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{L}x_0\right) - \frac{Lx_0}{n\pi} \cos\left(\frac{n\pi}{L}x_0\right), \quad (6.132)$$

$$\int_{x_0}^L (x-L) \sin\left(\frac{n\pi}{L}x\right) dx = \frac{L}{n\pi}(x_0-L) \cos\left(\frac{n\pi}{L}x_0\right) - \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi}{L}x_0\right). \quad (6.133)$$

Substituting (6.132) and (6.133) into equation (6.131), we find

$$a_n = 2\left(\frac{L}{n\pi}\right)^2 \frac{u_0}{x_0(L-x_0)} \sin\left(\frac{n\pi x_0}{L}\right). \quad (6.134)$$

Hence the formal solution is given by

$$u(x, t) = \frac{2L^2}{\pi^2} \frac{u_0}{x_0(L-x_0)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x_0}{L}\right) \cos\left(\frac{n\pi c}{L}t\right) \sin\left(\frac{n\pi}{L}x\right). \quad (6.135)$$

Note that the series (6.135) converges uniformly for all $x \in [0, L]$ and $t \geq 0$, but $u(x, t)$ is not a C^2 function (what happens when the series is differentiated term by term twice with respect to x and t ?). However, the partial sums $u_N(x, t)$ are smooth functions which satisfy the wave equation and the boundary conditions $u_N(0, t) = u_N(L, t) = 0$. Moreover, $u_N(x, 0) \rightarrow f(x)$ uniformly on $[0, L]$ as $N \rightarrow \infty$, so $u_N(x, t)$ is an approximate solution of our problem whose accuracy increases as N grows. \square

Neumann Boundary Conditions

A vibrating string with free ends satisfies the wave equation with Neumann boundary conditions

$$u_{tt} - c^2 u_{xx} = 0, \quad 0 < x < L, \quad (6.136)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L, \quad (6.137)$$

$$u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0. \quad (6.138)$$

Compatibility of the initial and boundary conditions implies that

$$f'(0) = f'(L) = 0, \quad g'(0) = g'(L) = 0. \quad (6.139)$$

The procedure for solving problem (6.136)–(6.138) is similar to the previous case with Dirichlet boundary conditions. We seek the solution in the form $u(x, t) = P(x)Q(t)$. From the boundary conditions $u_x(0, t) = P_x(0)Q(t) = 0$ and $u_x(L, t) = P_x(L)Q(t) = 0$, we obtain the following equations for the functions P and Q :

$$P''(x) + \lambda P(x) = 0, \quad P_x(0) = P_x(L) = 0, \quad (6.140)$$

$$Q''(t) + \lambda c^2 Q(t) = 0, \quad (6.141)$$

where $\lambda \in \mathbb{R}$ is the separation constant. The eigenvalues of the Sturm–Liouville problem (6.140) are discrete,

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \geq 0,$$

and the eigenfunctions $P_n(x)$ are

$$P_0 = B_0, \quad P_n(x) = B_n \cos\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \dots \quad (6.142)$$

For $\lambda = \lambda_n$, equation (6.141) has the solutions

$$Q_0 = C_0 + D_0 t, \quad (6.143)$$

$$Q_n(t) = C_n \cos\left(\frac{n\pi c}{L}t\right) + D_n \sin\left(\frac{n\pi c}{L}t\right), \quad n = 1, 2, 3, \dots \quad (6.144)$$

Thus we obtain a sequence of functions $u_n(x, t) = P_n(x)Q_n(t)$, which can be written in the form

$$u_0(x, t) = \frac{a_0 + b_0 t}{2}, \quad (6.145)$$

$$u_n(x, t) = \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \cos\left(\frac{n\pi}{L}x\right), \quad n = 1, 2, 3, \dots \quad (6.146)$$

for certain constants a_n and b_n . Therefore the solution $u(x, t) = \sum_{n=0}^{\infty} u_n(x, t)$ is given by

$$u(x, t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \cos\left(\frac{n\pi}{L}x\right), \quad (6.147)$$

and the Fourier coefficients a_n and b_n are determined by the initial conditions

$$u(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) = f(x), \quad 0 \leq x \leq L, \quad (6.148)$$

$$u_t(x, 0) = \frac{b_0}{2} + \sum_{n=1}^{\infty} \left(\frac{n\pi c}{L}\right) b_n \cos\left(\frac{n\pi}{L}x\right) = g(x), \quad 0 \leq x \leq L. \quad (6.149)$$

By taking even extensions of the functions f and g to the interval $[-L, L]$, we obtain

$$a_0 = \frac{2}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx, \quad (6.150)$$

$$b_0 = \frac{2}{L} \int_0^L g(x) dx, \quad b_n = \frac{2}{n\pi c} \int_0^L g(x) \cos\left(\frac{n\pi}{L}x\right) dx. \quad (6.151)$$

Thus the solution of the problem is completely determined.

6.3.2 Separation of Variables for the Nonhomogeneous Equation

The method of separation of variables can be adapted to solve the nonhomogeneous wave equation. We illustrate the procedure on the Neumann problem

$$u_{tt} - c^2 u_{xx} = F(x, t), \quad 0 < x < L, \quad t > 0, \quad (6.152)$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L, \quad (6.153)$$

$$u_x(0, t) = u_x(L, t) = 0, \quad t \geq 0. \quad (6.154)$$

From the previous discussion we know that if $F = 0$, then the solution is given by

$$u(x, t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right) \right] \cos\left(\frac{n\pi}{L}x\right). \quad (6.155)$$

This suggests that we seek the solution of the nonhomogeneous equation in the form

$$u(x, t) = \frac{1}{2}Q_0(t) + \sum_{n=1}^{\infty} Q_n(t) \cos\left(\frac{n\pi}{L}x\right), \quad (6.156)$$

where $Q_n(t)$ are unknown functions. Substituting (6.156) into the wave equation (6.152), we obtain

$$\frac{1}{2}Q_0''(t) + \sum_{n=1}^{\infty} \left[Q_n''(t) + \left(\frac{n\pi c}{L} \right)^2 Q_n(t) \right] \cos \left(\frac{n\pi}{L}x \right) = F(x, t). \quad (6.157)$$

If the function $F(x, t)$ can be expanded into a Fourier cosine series in the variable x , then the functions $Q_n(t)$ can be determined by comparing the coefficients of the two Fourier series. This will be possible if $F(x, t)$ satisfies the boundary conditions

$$F_x(0, t) = F_x(L, t) = 0, \quad t \geq 0. \quad (6.158)$$

Then

$$F(x, t) = \frac{1}{2}C_0(t) + \sum_{n=1}^{\infty} C_n(t) \cos \left(\frac{n\pi}{L}x \right), \quad 0 \leq x \leq L, \quad (6.159)$$

where the Fourier coefficients C_n depend on the variable t . Substituting this expression into equation (6.157), we obtain

$$\frac{1}{2} \left(Q_0''(t) - C_0(t) \right) + \sum_{n=1}^{\infty} \left[Q_n''(t) + \left(\frac{n\pi c}{L} \right)^2 Q_n(t) - C_n(t) \right] \cos \left(\frac{n\pi}{L}x \right) = 0. \quad (6.160)$$

This implies that the functions $Q_n(t)$ satisfy the differential equations

$$Q_0''(t) = C_0(t), \quad (6.161)$$

$$Q_n''(t) + \left(\frac{n\pi c}{L} \right)^2 Q_n(t) = C_n(t), \quad n \geq 1. \quad (6.162)$$

The solutions of these equations can be written as the sum of the solution of the corresponding homogeneous equation and a particular solution $Q_n^p(t)$:

$$Q_0(t) = a_0 + b_0 t + Q_0^p(t), \quad (6.163)$$

$$Q_n(t) = a_n \cos \left(\frac{n\pi c}{L}t \right) + b_n \sin \left(\frac{n\pi c}{L}t \right) + Q_n^p(t). \quad (6.164)$$

The functions $Q_n^p(t)$ are uniquely determined by the Fourier coefficients $C_n(t)$. Substituting the solutions for $Q_n(t)$ into equation (6.156), we find that

$$u(x, t) = u_h(x, t) + u_p(x, t), \quad (6.165)$$

where $u_h(x, t)$, given by (6.155), is the solution of the corresponding homogeneous equation, while

$$u_p(x, t) = \frac{1}{2}Q_0^p(t) + \sum_{n=1}^{\infty} Q_n^p(t) \cos\left(\frac{n\pi}{L}x\right) \quad (6.166)$$

is a particular solution. The coefficients a_n and b_n are determined by the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$.

Example 6.6 Determine the solution of the wave equation

$$u_{tt} - u_{xx} = \cos(2\pi x) \cos(2\pi t), \quad 0 < x < 1, \quad t > 0, \quad (6.167)$$

$$u(x, 0) = \cos^2(\pi x), \quad 0 \leq x \leq 1, \quad (6.168)$$

$$u_t(x, 0) = 2 \cos(2\pi x), \quad 0 \leq x \leq 1, \quad (6.169)$$

$$u_x(0, t) = u_x(1, t) = 0. \quad (6.170)$$

Solution. Here $L = c = 1$, so the general solution has the form

$$u(x, t) = \frac{1}{2}Q_0(t) + \sum_{n=1}^{\infty} Q_n(t) \cos(n\pi x). \quad (6.171)$$

Substituting (6.171) into equation (6.167), we obtain

$$\frac{1}{2}Q_0''(t) + \sum_{n=1}^{\infty} (Q_n''(t) + (n\pi)^2 Q_n(t)) \cos(n\pi x) = \cos(2\pi t) \cos(2\pi x). \quad (6.172)$$

For $n = 0, 2$, the functions Q_n satisfy

$$Q_0''(t) = 0, \quad (6.173)$$

$$Q_2''(t) + 4\pi^2 Q_2(t) = \cos(2\pi t), \quad (6.174)$$

while for $n \neq 0, 2$ we have

$$Q_n''(t) + (n\pi)^2 Q_n(t) = 0. \quad (6.175)$$

The solutions of equations (6.173) and (6.175) are

$$Q_0(t) = a_0 + b_0 t, \quad Q_n(t) = a_n \cos(n\pi t) + b_n \sin(n\pi t). \quad (6.176)$$

A particular solution of equation (6.174) has the form

$$Q_2^p(t) = \frac{t}{4\pi} \sin(2\pi t) \quad (6.177)$$

because the natural frequency and the frequency of the forced oscillation are both equal to 2π . Therefore

$$Q_2(t) = a_2 \cos(2\pi t) + b_2 \sin(2\pi t) + \frac{t}{4\pi} \sin(2\pi t). \quad (6.178)$$

Hence the solution of the equation is given by the series

$$u(x, t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \cos(n\pi x) + \frac{t}{4\pi} \sin(2\pi t) \cos(2\pi x), \quad (6.179)$$

where the coefficients a_n and b_n are determined by the initial conditions. From condition (6.168) we obtain

$$u(x, 0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) = \cos^2(\pi x). \quad (6.180)$$

Using the identity $\cos^2(\pi x) = \frac{1}{2} + \frac{1}{2} \cos(2\pi x)$ and comparing Fourier coefficients in the above equation, we conclude that

$$a_0 = 1, \quad a_2 = \frac{1}{2}, \quad a_n = 0, \quad n \neq 0, 2. \quad (6.181)$$

Similarly, from condition (6.169) it follows that

$$u_t(x, 0) = \frac{b_0}{2} + \sum_{n=1}^{\infty} n\pi b_n \cos(n\pi x) = 2 \cos(2\pi x), \quad (6.182)$$

from which we obtain

$$b_2 = \frac{1}{\pi}, \quad b_n = 0, \quad n \neq 2. \quad (6.183)$$

Substituting the Fourier coefficients a_n and b_n into the expression for $u(x, t)$, we find the solution of the problem

$$u(x, t) = \frac{1}{2} + \left(\frac{1}{2} \cos(2\pi t) + \frac{4+t}{4\pi} \sin(2\pi t) \right) \cos(2\pi x). \quad (6.184)$$

□

Chapter 7

Laplace's Equation

One of the most important examples of elliptic equations is Laplace's equation, named after the French mathematician and physicist Pierre-Simon de Laplace (1749–1827), who made major contributions to the development of potential theory. Laplace's equation has applications in electrostatics, gravitational theory, fluid mechanics, and many other problems in physics and engineering. In this chapter we study Laplace's equation in two dimensions

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \Omega, \quad (7.1)$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain. The differential operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (7.2)$$

is called the Laplace operator. Laplace's equation is a special case of Poisson's equation

$$\Delta u = f(x, y), \quad (x, y) \in \Omega \quad (7.3)$$

corresponding to $f = 0$. In the first part of the chapter we consider general properties of Laplace's equation, the most important of which are the maximum principles and the mean value property. By the method of separation of variables we will construct solutions of Laplace's equation for rectangular and circular domains. Finally, we will derive the solution of Laplace's equation in the form of Poisson's formula and extend the method of separation of variables to Poisson's equation.

7.1 General Properties of Laplace's Equation

A function $u \in C^2(\Omega)$ satisfying Laplace's equation in a domain $\Omega \subseteq \mathbb{R}^2$ is called a *harmonic* function in Ω . Harmonic functions arise naturally in the theory of functions of a complex variable. If $f(z) = u(x, y) + iv(x, y)$ is an analytic function in $\Omega \subseteq \mathbb{C}$, then u and v satisfy the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$. Hence

$$u_{xx} + u_{yy} = \frac{\partial}{\partial x}(v_y) - \frac{\partial}{\partial y}(v_x) = 0. \quad (7.4)$$

Similarly, one shows that v satisfies $v_{xx} + v_{yy} = 0$. Thus, the real and imaginary parts of an analytic function $f = u + iv$ are harmonic functions. For example,

$$e^z = e^x(\cos(y) + i \sin(y)) \quad (7.5)$$

is an analytic function in \mathbb{C} . It is easy to verify that $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$ are harmonic functions in \mathbb{R}^2 .

In the following we assume that the boundary of the bounded domain Ω is the union

$$\partial\Omega = \bigcup_{i=1}^n C_i$$

of piecewise smooth simple closed curves C_1, C_2, \dots, C_n . The unit normal vector \vec{n} is defined at every point of the boundary $\partial\Omega$, except possibly at the points where the curves C_i meet.

Definition 7.1 *Let u be a solution of Poisson's equation (7.3) on the domain Ω , and let g be a function defined on $\partial\Omega$. We say that the function u satisfies*

(i) *the Dirichlet condition on Ω if*

$$u(x, y) = g(x, y) \quad \text{for every } (x, y) \in \partial\Omega, \quad (7.6)$$

(ii) *the Neumann condition on Ω if*

$$\frac{\partial u}{\partial \vec{n}}(x, y) = g(x, y) \quad \text{for every } (x, y) \in \partial\Omega, \quad (7.7)$$

where \vec{n} is the outward-pointing unit normal vector to $\partial\Omega$, and $\partial u / \partial \vec{n} = \nabla u \cdot \vec{n}$ is the directional derivative of u in the direction of the vector \vec{n} .

If a solution of Poisson's equation satisfies the Neumann condition, then the functions f and g must satisfy the so-called consistency condition.

Lemma 7.1 *Let Ω be a bounded domain in \mathbb{R}^2 . A necessary condition for the existence of a solution of the Neumann problem*

$$u_{xx} + u_{yy} = f(x, y), \quad (x, y) \in \Omega, \quad (7.8)$$

$$\frac{\partial u}{\partial \vec{n}}(x, y) = g(x, y), \quad (x, y) \in \partial\Omega, \quad (7.9)$$

is the consistency condition

$$\int_{\partial\Omega} g \, ds = \iint_{\Omega} f \, dx dy. \quad (7.10)$$

Proof. Using the vector identity for the Laplace operator

$$\Delta u = \nabla \cdot (\nabla u), \quad (7.11)$$

Poisson's equation can be written in the form

$$\nabla \cdot (\nabla u) = f. \quad (7.12)$$

If \vec{G} is a C^1 vector field on Ω , then by Gauss's theorem

$$\iint_{\Omega} \nabla \cdot \vec{G} \, dx dy = \int_{\partial\Omega} (\vec{G} \cdot \vec{n}) \, ds, \quad (7.13)$$

where \vec{n} is the outward-pointing unit normal vector to $\partial\Omega$. Integrating equation (7.12) over Ω and applying Gauss's theorem, we obtain

$$\begin{aligned} \iint_{\Omega} f \, dx dy &= \iint_{\Omega} \nabla \cdot (\nabla u) \, dx dy \\ &= \int_{\partial\Omega} (\nabla u \cdot \vec{n}) \, ds = \int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} \, ds = \int_{\partial\Omega} g \, ds. \end{aligned}$$

■

Note that in the case when u is a solution of Laplace's equation ($f = 0$), a necessary condition for the existence of a solution is

$$\int_{\partial\Omega} g \, ds = 0. \quad (7.14)$$

In other words, the integral of the normal derivative of a harmonic function u over $\partial\Omega$ vanishes:

$$\int_{\partial\Omega} \frac{\partial u}{\partial \vec{n}} ds = 0. \quad (7.15)$$

If the consistency condition is not satisfied, then the Neumann problem is not solvable.

The following result shows that a harmonic function on a bounded domain attains its maximum and minimum on the boundary of that domain. This conclusion is analogous to the maximum principle for the heat equation.

Theorem 7.1 (Weak Maximum Principle) *Let Ω be a bounded domain in \mathbb{R}^2 and let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a harmonic function on Ω . Then*

$$\max_{(x,y) \in \bar{\Omega}} u(x,y) = u(x',y') \quad (7.16)$$

for some point $(x',y') \in \partial\Omega$. In other words, the function u attains its maximum on $\bar{\Omega}$ at some point of the boundary $\partial\Omega$.

Proof. Let $\varepsilon > 0$. Define the function

$$v(x,y) = u(x,y) + \varepsilon(x^2 + y^2), \quad (x,y) \in \bar{\Omega}. \quad (7.17)$$

The function v is continuous on the compact set $\bar{\Omega}$, and therefore v attains its maximum at some point $(x_0, y_0) \in \bar{\Omega}$. If (x_0, y_0) is an interior point of $\bar{\Omega}$, that is, $(x_0, y_0) \in \Omega$, then $v(x_0, y_0)$ is a local maximum, and hence

$$v_{xx}(x_0, y_0) \leq 0 \quad \text{and} \quad v_{yy}(x_0, y_0) \leq 0. \quad (7.18)$$

However,

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + 4\varepsilon = 4\varepsilon > 0 \quad (7.19)$$

at every point $(x,y) \in \Omega$, so we conclude that $(x_0, y_0) \notin \Omega$. Thus v attains its maximum on the boundary $\partial\Omega$, and therefore

$$\max_{(x,y) \in \bar{\Omega}} v(x,y) = v(x_0, y_0) \quad \text{for some point } (x_0, y_0) \in \partial\Omega. \quad (7.20)$$

Let

$$M = \max_{(x,y) \in \partial\Omega} u(x,y) \quad \text{and} \quad K = \max_{(x,y) \in \partial\Omega} (x^2 + y^2). \quad (7.21)$$

Now from (7.20) and (7.21) it follows that for every $(x, y) \in \Omega$ we have

$$v(x, y) \leq \max_{(x,y) \in \partial\Omega} v(x, y) = \max_{(x,y) \in \partial\Omega} (u(x, y) + \varepsilon(x^2 + y^2)) = M + \varepsilon K. \quad (7.22)$$

Therefore, for every $(x, y) \in \Omega$,

$$u(x, y) = v(x, y) - \varepsilon(x^2 + y^2) \leq M + \varepsilon K - \varepsilon(x^2 + y^2) \leq M + \varepsilon K. \quad (7.23)$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that

$$u(x, y) \leq M \quad \text{for every } (x, y) \in \Omega. \quad (7.24)$$

This implies

$$\max_{(x,y) \in \bar{\Omega}} u(x, y) = M = \max_{(x,y) \in \partial\Omega} u(x, y). \quad (7.25)$$

The function u is continuous on the compact set $\bar{\Omega}$, so there exists $(x', y') \in \partial\Omega$ such that

$$\max_{(x,y) \in \bar{\Omega}} u(x, y) = u(x', y'). \quad (7.26)$$

■

Note that the weak maximum principle does not exclude the possibility that $u(x, y)$ attains its maximum at some point $(x, y) \in \Omega$, for example if $u(x, y)$ is constant.

If u is harmonic on Ω , then $v = -u$ is also harmonic on Ω . Since

$$\max_{(x,y) \in \bar{\Omega}} v(x, y) = \min_{(x,y) \in \bar{\Omega}} u(x, y),$$

the previous theorem immediately implies the following result.

Theorem 7.2 (Weak Minimum Principle) *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a harmonic function on a bounded domain $\Omega \subseteq \mathbb{R}^2$. Then there exists $(x', y') \in \partial\Omega$ such that*

$$\min_{(x,y) \in \bar{\Omega}} u(x, y) = u(x', y'). \quad (7.27)$$

An immediate consequence of the maximum and minimum principles is the following.

Corollary 7.1 *If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a harmonic function on a bounded domain $\Omega \subseteq \mathbb{R}^2$ and $u(x, y) = 0$ for every $(x, y) \in \partial\Omega$, then $u = 0$.*

Proof. By the weak maximum and minimum principles we have

$$0 = \min_{(x,y) \in \partial\Omega} u(x,y) \leq u(x,y) \leq \max_{(x,y) \in \partial\Omega} u(x,y) = 0 \quad (7.28)$$

for every $(x,y) \in \bar{\Omega}$, and hence $u = 0$. ■

One of the important consequences of the maximum principle is the uniqueness of the solution of the Dirichlet problem for Poisson's equation.

Theorem 7.3 *Let Ω be a bounded domain in \mathbb{R}^2 . Then there exists at most one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of the Dirichlet problem*

$$\Delta u(x,y) = f(x,y), \quad (x,y) \in \Omega, \quad (7.29)$$

$$u(x,y) = g(x,y), \quad (x,y) \in \partial\Omega. \quad (7.30)$$

Proof. Assume that there exist two solutions u_1 and u_2 of the problem (7.29)–(7.30). Then $u = u_1 - u_2$ is a harmonic function satisfying

$$\Delta u(x,y) = 0, \quad (x,y) \in \Omega, \quad (7.31)$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega. \quad (7.32)$$

By Corollary 7.1, we have $u = 0$, which implies $u_1 = u_2$. ■

Let us emphasize that the boundedness of the domain Ω is an important assumption in Theorem 7.3. Consider the Dirichlet problem on the unbounded domain $\Omega = \{(x,y) \mid x^2 + y^2 > 4\}$:

$$\Delta u(x,y) = 0, \quad (x,y) \in \Omega, \quad (7.33)$$

$$u(x,y) = 1, \quad x^2 + y^2 = 4. \quad (7.34)$$

It is easy to verify that the functions $u_1(x,y) = 1$ and $u_2(x,y) = \frac{1}{\ln(2)} \ln \sqrt{x^2 + y^2}$ are solutions of the problem (7.33)–(7.34), so the solution of Laplace's equation on Ω is not unique.

The maximum principle also implies that solutions of the Dirichlet problem for Poisson's equation are stable with respect to the boundary conditions. In other words, small changes in the boundary conditions result in small changes in the solution of Poisson's equation.

Theorem 7.4 *Let Ω be a bounded domain in \mathbb{R}^2 , and let $u_1, u_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ be solutions of Poisson's equation*

$$\Delta u_1(x, y) = f(x, y), \quad \Delta u_2(x, y) = f(x, y), \quad (x, y) \in \Omega, \quad (7.35)$$

which satisfy the boundary conditions

$$u_1(x, y) = g_1(x, y), \quad u_2(x, y) = g_2(x, y), \quad (x, y) \in \partial\Omega, \quad (7.36)$$

where g_1 and g_2 are continuous functions on $\partial\Omega$. If

$$\max_{\partial\Omega} |g_1 - g_2| < \varepsilon, \quad (7.37)$$

then

$$\max_{\bar{\Omega}} |u_1 - u_2| < \varepsilon. \quad (7.38)$$

Proof. Define $u = u_1 - u_2$. Then u is a solution of Laplace's equation $\nabla^2 u = 0$ in Ω satisfying the boundary condition $u(x, y) = g_1(x, y) - g_2(x, y)$, $(x, y) \in \partial\Omega$. On the boundary of the domain we have

$$|u(x, y)| = |g_1(x, y) - g_2(x, y)| < \varepsilon, \quad (x, y) \in \partial\Omega, \quad (7.39)$$

which implies

$$-\varepsilon < \min_{\partial\Omega} u \quad \text{and} \quad \max_{\partial\Omega} u < \varepsilon. \quad (7.40)$$

By the weak maximum and minimum principles, we have

$$-\varepsilon < \min_{\partial\Omega} u \leq u(x, y) \leq \max_{\partial\Omega} u < \varepsilon, \quad \forall (x, y) \in \Omega. \quad (7.41)$$

Therefore,

$$|u(x, y)| = |u_1(x, y) - u_2(x, y)| < \varepsilon \quad \text{for every } (x, y) \in \bar{\Omega}. \quad (7.42)$$

■

Theorem 7.5 (Mean Value Property) *Let u be a harmonic function in a domain Ω (which need not be bounded), and let $\bar{K}_r(x_0, y_0) \subset \Omega$ be a closed disk of radius $r > 0$ centered at $(x_0, y_0) \in \Omega$. Then*

$$u(x_0, y_0) = \frac{1}{2\pi r} \int_{C_r} u \, ds \quad (7.43)$$

where C_r is the circle of radius $r > 0$ centered at (x_0, y_0) .

According to this property, the value of a harmonic function at the center of a circle is equal to the average value of the function over the circle.

Proof. Let $0 < \rho \leq r$ and let C_ρ be the circle of radius ρ centered at (x_0, y_0) . Define the function

$$V(\rho) = \frac{1}{2\pi\rho} \int_{C_\rho} u \, ds = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + \rho \cos(\varphi), y_0 + \rho \sin(\varphi)) \, d\varphi. \quad (7.44)$$

The function $V(\rho)$ has a removable discontinuity at $\rho = 0$ because

$$\lim_{\rho \rightarrow 0^+} V(\rho) = \lim_{\rho \rightarrow 0^+} \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + \rho \cos(\varphi), y_0 + \rho \sin(\varphi)) \, d\varphi = u(x_0, y_0). \quad (7.45)$$

Let us show that $V(\rho)$ is constant. Differentiating, we obtain

$$\begin{aligned} V'(\rho) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \rho} u(x_0 + \rho \cos(\varphi), y_0 + \rho \sin(\varphi)) \, d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} [u_x(x_0 + \rho \cos(\varphi), y_0 + \rho \sin(\varphi)) \cos(\varphi) \\ &\quad + u_y(x_0 + \rho \cos(\varphi), y_0 + \rho \sin(\varphi)) \sin(\varphi)] \, d\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial \vec{n}} \, d\varphi = \frac{1}{2\pi\rho} \int_{C_\rho} \frac{\partial u}{\partial \vec{n}} \, ds \end{aligned} \quad (7.46)$$

where $\frac{\partial u}{\partial \vec{n}} = \nabla u \cdot \vec{n}$ is the directional derivative in the direction of the unit radial vector $\vec{n} = \cos(\varphi)\vec{i} + \sin(\varphi)\vec{j}$. By equation (7.15), we have

$$\int_{C_\rho} \frac{\partial u}{\partial \vec{n}} \, ds = 0 \quad (7.47)$$

since u is harmonic in the disk $K_\rho(x_0, y_0)$. Hence $V'(\rho) = 0$, which implies that $V(\rho)$ is constant. It follows that

$$V(r) = \lim_{\rho \rightarrow 0^+} V(\rho) = u(x_0, y_0), \quad (7.48)$$

which implies

$$\frac{1}{2\pi r} \int_{C_r} u \, ds = u(x_0, y_0). \quad (7.49)$$

■

The mean value property characterizes harmonic functions, since the converse of this theorem is also true.

Theorem 7.6 *Assume that a function $u \in C^2(\Omega)$ satisfies the mean value property at every point of the domain Ω . Then u is a harmonic function in Ω .*

Proof. Assume that there exists $(x_0, y_0) \in \Omega$ such that $\Delta u(x_0, y_0) \neq 0$. Without loss of generality, we may assume that $\Delta u(x_0, y_0) > 0$. Since Δu is continuous on Ω , there exists $r > 0$ such that $\Delta u(x, y) > 0$ in the disk K_r of radius r centered at (x_0, y_0) . Let C_r be the boundary of the disk K_r . By Gauss's theorem (see (7.13)), it follows that

$$\begin{aligned} 0 &< \frac{1}{2\pi} \iint_{K_r} \Delta u \, dx dy = \frac{1}{2\pi} \int_{C_r} \frac{\partial u}{\partial \vec{n}} \, ds = \frac{1}{2\pi} \int_0^{2\pi} (\nabla u \cdot \vec{n}) \, r d\varphi \\ &= \frac{r}{2\pi} \int_0^{2\pi} \left[u_x(x_0 + r \cos(\varphi), y_0 + r \sin(\varphi)) \cos(\varphi) \right. \\ &\quad \left. + u_y(x_0 + r \cos(\varphi), y_0 + r \sin(\varphi)) \sin(\varphi) \right] d\varphi \\ &= \frac{r}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial r} u(x_0 + r \cos(\varphi), y_0 + r \sin(\varphi)) \, d\varphi \\ &= r \frac{\partial}{\partial r} \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos(\varphi), y_0 + r \sin(\varphi)) \, d\varphi. \end{aligned} \quad (7.50)$$

where $\vec{n} = \cos(\varphi)\vec{i} + \sin(\varphi)\vec{j}$ is the unit normal vector on C_r . By assumption, u satisfies the mean value property in Ω , so

$$\frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos(\varphi), y_0 + r \sin(\varphi)) \, d\varphi = u(x_0, y_0). \quad (7.51)$$

This leads to a contradiction, because from equation (7.50) we obtain

$$r \frac{\partial}{\partial r} \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos(\varphi), y_0 + r \sin(\varphi)) \, d\varphi = r \frac{\partial}{\partial r} u(x_0, y_0) = 0. \quad (7.52)$$

Therefore, $\Delta u(x_0, y_0) = 0$ for every $(x_0, y_0) \in \Omega$, and we conclude that u is a harmonic function in Ω . ■

The following result can be proved from the mean value property.

Theorem 7.7 (Strong Maximum Principle) *Let u be a harmonic function in a domain Ω (which need not be bounded). If u attains a minimum or a maximum in the interior of Ω , then u is constant.*

The strong maximum principle guarantees that nonconstant harmonic functions cannot attain their minimum or maximum value in the interior of the domain Ω . If Ω is a bounded domain, then we know that the minimum or maximum is attained at some point of the boundary $\partial\Omega$. However, if Ω is unbounded, then a harmonic function u need not attain its maximum or minimum on $\bar{\Omega} = \Omega \cup \partial\Omega$. For example, $u(x, y) = \ln(x^2 + y^2)$ is a harmonic function in the domain $\Omega = \{(x, y) \mid x^2 + y^2 \geq 1\}$ and $u(x, y) = 0$ at every point $(x, y) \in \partial\Omega$, but u does not attain its maximum on $\bar{\Omega}$.

7.2 Separation of Variables for Laplace's Equation

The method of separation of variables can be applied to Laplace's equation when the domain Ω has a certain symmetry. In this section we study methods for solving Laplace's equation on rectangular and circular domains. A formal solution of Laplace's equation is given in the form of a series $u = \sum_{n=1}^{\infty} u_n$, where each u_n is a harmonic function in Ω . It is therefore necessary to know under which conditions the series $\sum_{n=1}^{\infty} u_n$ converges to a harmonic function, that is, which conditions guarantee that $u = \sum_{n=1}^{\infty} u_n$ is a classical solution of Laplace's equation.

Theorem 7.8 *Let Ω be a bounded domain in \mathbb{R}^2 . Let $u = \sum_{n=1}^{\infty} u_n$ be a formal solution of the Dirichlet problem*

$$\Delta u(x, y) = 0, \quad (x, y) \in \Omega, \quad (7.53)$$

$$u(x, y) = g(x, y), \quad (x, y) \in \partial\Omega, \quad (7.54)$$

where g is a continuous function on $\partial\Omega$ and each $u_n \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω for every $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} u_n$ converges uniformly to the function g on $\partial\Omega$, then $\sum_{n=1}^{\infty} u_n$ converges uniformly on $\bar{\Omega}$, and u is a classical solution of the problem (7.53)–(7.54).

Proof. Define the partial sums $s_n = \sum_{k=1}^n u_k$. Then $s_n \in C^2(\Omega) \cap C(\bar{\Omega})$ is a sequence of harmonic functions which converges uniformly to g on $\partial\Omega$. Let $\varepsilon > 0$. By the Cauchy criterion for uniform convergence, there exists $n_0 \in \mathbb{N}$ such that

$$n, m > n_0 \quad \Rightarrow \quad \sup_{(x, y) \in \partial\Omega} |s_n(x, y) - s_m(x, y)| < \varepsilon. \quad (7.55)$$

Since $s_n - s_m \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic in Ω , the weak maximum principle implies that

$$\sup_{(x,y) \in \bar{\Omega}} |s_n(x,y) - s_m(x,y)| < \varepsilon. \quad (7.56)$$

By the Cauchy criterion, the sequence $\{s_n\}$ converges uniformly on $\bar{\Omega}$, and hence the series $\sum_{k=1}^{\infty} u_k$ converges uniformly on $\bar{\Omega}$.

Let us show that $u = \sum_{k=1}^{\infty} u_k$ is a harmonic function in Ω . Let $(x_0, y_0) \in \Omega$ and let $K_r(x_0, y_0) \subset \Omega$ be a disk of radius $r > 0$ centered at (x_0, y_0) . The functions u_n are harmonic in Ω , so they satisfy the mean value property

$$u_n(x_0, y_0) = \frac{1}{2\pi r} \int_{\partial K_r} u_n ds \quad (7.57)$$

for every $n \in \mathbb{N}$. Hence

$$\begin{aligned} u(x_0, y_0) &= \sum_{n=1}^{\infty} u_n(x_0, y_0) = \sum_{n=1}^{\infty} \frac{1}{2\pi r} \int_{\partial K_r} u_n ds \\ &= \frac{1}{2\pi r} \int_{\partial K_r} \sum_{n=1}^{\infty} u_n ds = \frac{1}{2\pi r} \int_{\partial K_r} u ds \end{aligned} \quad (7.58)$$

where the integral and the series may be interchanged because of uniform convergence. Thus the function u satisfies the mean value property at every point $(x_0, y_0) \in \Omega$, and therefore by Theorem 7.6 u is harmonic in Ω . Furthermore, u satisfies the boundary condition $u(x, y) = \sum_{k=1}^{\infty} u_k(x, y) = g(x, y)$, $(x, y) \in \partial\Omega$, so u is a classical solution of the problem (7.53)–(7.54). ■

7.2.1 Rectangular Domains

In this section we develop the method of separation of variables for Laplace's equation on a rectangular domain. By translating the coordinate system, we may assume that the domain is given by $\Omega = (0, b) \times (0, d)$. Consider Laplace's equation

$$\Delta u(x, y) = 0, \quad (x, y) \in \Omega \quad (7.59)$$

with Dirichlet boundary conditions

$$u(x, 0) = h(x), \quad u(x, d) = k(x), \quad 0 \leq x \leq b, \quad (7.60)$$

$$u(0, y) = f(y), \quad u(b, y) = g(y), \quad 0 \leq y \leq d. \quad (7.61)$$

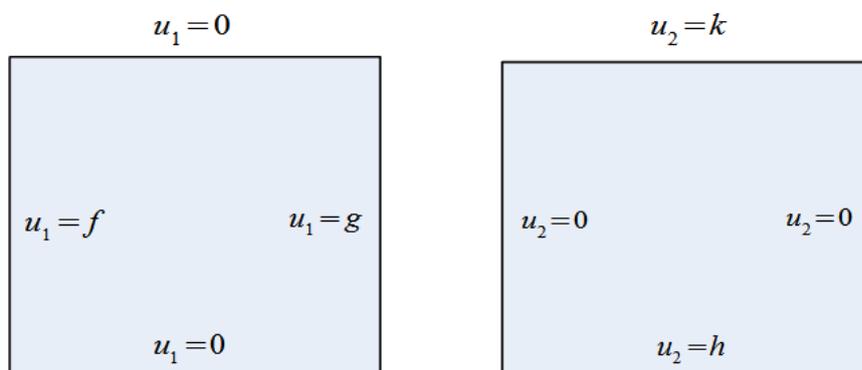


Figure 7.1:

To solve the problem, one must associate with Laplace's equation the appropriate Sturm–Liouville problem that provides the basis functions with respect to which the solution u is expanded in a series. Recall that a Sturm–Liouville problem requires homogeneous boundary conditions. Therefore the solution u must be decomposed as a sum $u = u_1 + u_2$, where u_1 and u_2 are harmonic functions satisfying the following boundary conditions (see Figure 7.1):

$$u_1(x, 0) = 0, \quad u_1(x, d) = 0, \quad 0 \leq x \leq b, \quad (7.62)$$

$$u_1(0, y) = f(y), \quad u_1(b, y) = g(y), \quad 0 \leq y \leq d, \quad (7.63)$$

$$u_2(x, 0) = h(x), \quad u_2(x, d) = k(x), \quad 0 \leq x \leq b, \quad (7.64)$$

$$u_2(0, y) = 0, \quad u_2(b, y) = 0, \quad 0 \leq y \leq d. \quad (7.65)$$

The boundary conditions in Theorem 7.8 are defined by a continuous function on $\partial\Omega$. In order for the boundary conditions for the functions u_1 and u_2 to be continuous, we must assume that the functions f , g , h , and k satisfy the compatibility conditions

$$f(0) = f(d) = 0, \quad g(0) = g(d) = 0, \quad (7.66)$$

$$h(0) = h(b) = 0, \quad k(0) = k(b) = 0. \quad (7.67)$$

Clearly, $u = u_1 + u_2$ satisfies the boundary conditions (7.60)–(7.61), so by Theorem 7.3 the function $u = u_1 + u_2$ is the unique solution of the problem (7.59)–(7.61).

Let us show how to determine the function u_1 . We seek the solution in the separated form $u_1(x, y) = P(x)Q(y)$. From Laplace's equation we obtain

$$\frac{P''(x)}{P(x)} = -\frac{Q''(y)}{Q(y)} = \lambda \quad (7.68)$$

for some $\lambda \in \mathbb{R}$. Thus the functions P and Q satisfy the equations

$$P''(x) - \lambda P(x) = 0, \quad 0 < x < b, \quad (7.69)$$

$$Q''(y) + \lambda Q(y) = 0, \quad 0 < y < d. \quad (7.70)$$

The boundary conditions (7.62) imply

$$Q(0) = Q(d) = 0, \quad (7.71)$$

so the function Q satisfies the associated Sturm–Liouville problem (7.70)–(7.71). The eigenvalues and eigenfunctions are

$$\lambda_n = \left(\frac{n\pi}{d}\right)^2, \quad Q_n(y) = \sin\left(\frac{n\pi}{d}y\right), \quad n \in \mathbb{N}. \quad (7.72)$$

Now the equation for the function P has the form

$$P''(x) - \left(\frac{n\pi}{d}\right)^2 P(x) = 0, \quad 0 < x < b. \quad (7.73)$$

Because of the boundary conditions on the sides of the rectangle $x = 0$ and $x = b$, we write the general solution as

$$P_n(x) = A_n \sinh\left(\frac{n\pi}{d}x\right) + B_n \sinh\left(\frac{n\pi}{d}(x - b)\right), \quad n \in \mathbb{N}. \quad (7.74)$$

Note that the hyperbolic functions $\sinh(n\pi x/d)$ and $\sinh(n\pi(x - b)/d)$ are linearly independent solutions of equation (7.73). By the principle of superposition, the formal solution u_1 is given by the series

$$u_1(x, y) = \sum_{n=1}^{\infty} \left[A_n \sinh\left(\frac{n\pi}{d}x\right) + B_n \sinh\left(\frac{n\pi}{d}(x - b)\right) \right] \sin\left(\frac{n\pi}{d}y\right). \quad (7.75)$$

Substituting (7.75) into the boundary conditions (7.63), we obtain

$$u_1(0, y) = \sum_{n=1}^{\infty} -B_n \sinh\left(\frac{n\pi b}{d}\right) \sin\left(\frac{n\pi}{d}y\right) = f(y), \quad 0 \leq y \leq d, \quad (7.76)$$

$$u_1(b, y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{d}\right) \sin\left(\frac{n\pi}{d}y\right) = g(y), \quad 0 \leq y \leq d. \quad (7.77)$$

Define the coefficients

$$\alpha_n = A_n \sinh\left(\frac{n\pi b}{d}\right), \quad \beta_n = -B_n \sinh\left(\frac{n\pi b}{d}\right). \quad (7.78)$$

Equations (7.76) and (7.77) represent the expansions of the functions f and g into Fourier sine series on the interval $[0, d]$:

$$g(y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi}{d}y\right), \quad f(y) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi}{d}y\right), \quad 0 \leq y \leq d, \quad (7.79)$$

where the Fourier coefficients are given by

$$\alpha_n = \frac{2}{d} \int_0^d g(y) \sin\left(\frac{n\pi}{d}y\right) dy, \quad \beta_n = \frac{2}{d} \int_0^d f(y) \sin\left(\frac{n\pi}{d}y\right) dy. \quad (7.80)$$

Now from equation (7.78) we obtain

$$A_n = \frac{2}{d \sinh\left(\frac{n\pi b}{d}\right)} \int_0^d g(y) \sin\left(\frac{n\pi}{d}y\right) dy, \quad (7.81)$$

$$B_n = -\frac{2}{d \sinh\left(\frac{n\pi b}{d}\right)} \int_0^d f(y) \sin\left(\frac{n\pi}{d}y\right) dy. \quad (7.82)$$

By a similar calculation one can show that the solution for $u_2(x, y)$ is given by

$$u_2(x, y) = \sum_{n=1}^{\infty} \left[C_n \sinh\left(\frac{n\pi}{b}y\right) + D_n \sinh\left(\frac{n\pi}{b}(y-d)\right) \right] \sin\left(\frac{n\pi}{b}x\right) \quad (7.83)$$

where

$$C_n = \frac{2}{b \sinh\left(\frac{n\pi d}{b}\right)} \int_0^b k(x) \sin\left(\frac{n\pi}{b}x\right) dx, \quad (7.84)$$

$$D_n = -\frac{2}{b \sinh\left(\frac{n\pi d}{b}\right)} \int_0^b h(x) \sin\left(\frac{n\pi}{b}x\right) dx. \quad (7.85)$$

In the above procedure we assumed that the boundary functions f , g , h , and k satisfy the compatibility conditions (7.66)–(7.67). In applications this assumption is often not justified, and therefore the procedure must be modified in order to include boundary conditions that do not vanish at the vertices of the rectangle. This can

be done by adding harmonic polynomials to the functions that define the boundary conditions. Consider the Dirichlet problem

$$\Delta u(x, y) = 0, \quad (x, y) \in \Omega, \quad (7.86)$$

$$u(x, y) = G(x, y), \quad (x, y) \in \partial\Omega. \quad (7.87)$$

where $G(x, y)$ is a continuous function on the boundary of the rectangle $\Omega = (0, b) \times (0, d)$. Our task is to transform the problem (7.86)–(7.87) into a Dirichlet problem with continuous boundary conditions that vanish at the vertices of the rectangle Ω . Decompose the function u as a sum $u(x, y) = v(x, y) + P_2(x, y)$, where $P_2(x, y)$ is a harmonic polynomial of degree two. The general form of the polynomial P_2 is

$$P_2(x, y) = a_1(x^2 - y^2) + a_2xy + a_3x + a_4y + a_5, \quad (7.88)$$

where a_1, \dots, a_5 are arbitrary coefficients. The function v is harmonic in Ω and at the vertices of the rectangle satisfies

$$v(0, 0) = G(0, 0) - P_2(0, 0), \quad (7.89)$$

$$v(0, d) = G(0, d) - P_2(0, d), \quad (7.90)$$

$$v(b, 0) = G(b, 0) - P_2(b, 0), \quad (7.91)$$

$$v(b, d) = G(b, d) - P_2(b, d). \quad (7.92)$$

If we choose the coefficients of the polynomial P_2 so that

$$P_2(0, 0) = G(0, 0), \quad P_2(0, d) = G(0, d), \quad (7.93)$$

$$P_2(b, 0) = G(b, 0), \quad P_2(b, d) = G(b, d), \quad (7.94)$$

then the function $v(x, y)$ satisfies the Dirichlet problem

$$\nabla^2 v(x, y) = 0, \quad (x, y) \in \Omega, \quad (7.95)$$

$$v(x, y) = \tilde{G}(x, y), \quad (x, y) \in \partial\Omega, \quad (7.96)$$

where $\tilde{G}(x, y) = G(x, y) - P_2(x, y)$ is a continuous function on $\partial\Omega$ that satisfies the compatibility conditions, since it vanishes at the vertices of the rectangle. The function v can then be determined by the procedure described above. In this way we obtain the solution of the problem (7.86)–(7.87) in the form

$$u(x, y) = v(x, y) + P_2(x, y). \quad (7.97)$$

Note that the conditions (7.93)–(7.94) give four equations for five unknowns a_1, \dots, a_5 , which yields infinitely many harmonic polynomials $P_2(x, y)$. This is not in contradiction with the fact that the solution of the Dirichlet problem is unique, because the choice of the polynomial P_2 changes the boundary conditions for the function v , and hence also the function v itself. However, the solution (7.97) remains unchanged.

Example 7.1 Determine the solution of Laplace's equation on the rectangle $\Omega = (0, 1) \times (0, 1)$ with boundary conditions

$$u(x, 0) = 1 + \sin(\pi x), \quad u(0, y) = 1 + y, \quad (7.98)$$

$$u(x, 1) = 2, \quad u(1, y) = 1 + y. \quad (7.99)$$

Solution. Note that the boundary conditions are given by a continuous function on the boundary of the rectangle, but they do not satisfy the compatibility condition. Let $u(x, y) = v(x, y) + P_2(x, y)$, where $P_2(x, y) = a_1(x^2 - y^2) + a_2xy + a_3x + a_4y + a_5$. The function $v(x, y)$ satisfies the boundary conditions

$$v(x, 0) = 1 + \sin(\pi x) - P_2(x, 0), \quad v(0, y) = 1 + y - P_2(0, y), \quad (7.100)$$

$$v(x, 1) = 2 - P_2(x, 1), \quad v(1, y) = 1 + y - P_2(1, y). \quad (7.101)$$

We choose the polynomial $P_2(x, y)$ so that the boundary conditions for $v(x, y)$ vanish at the vertices of the rectangle:

$$v(0, 0) = 1 - P_2(0, 0) = 0, \quad (7.102)$$

$$v(0, 1) = 2 - P_2(0, 1) = 0, \quad (7.103)$$

$$v(1, 0) = 1 - P_2(1, 0) = 0, \quad (7.104)$$

$$v(1, 1) = 2 - P_2(1, 1) = 0. \quad (7.105)$$

It follows that the coefficients of the polynomial satisfy the system of equations

$$a_5 = 1, \quad -a_1 + a_4 + a_5 = 2, \quad a_1 + a_3 + a_5 = 1, \quad a_2 + a_3 + a_4 + a_5 = 2. \quad (7.106)$$

One of the coefficients may be chosen arbitrarily. If we choose $a_1 = 0$, then $a_2 = a_3 = 0$ and $a_4 = 1$, so we obtain

$$P_2(x, y) = 1 + y. \quad (7.107)$$

The boundary conditions for the function $v(x, y)$ are now

$$v(x, 0) = \sin(\pi x), \quad v(0, y) = 0, \quad (7.108)$$

$$v(x, 1) = 0, \quad v(1, y) = 0. \quad (7.109)$$

The boundary conditions on the sides of the rectangle $x = 0$ and $x = 1$ are homogeneous, so the method of separation of variables can be applied to the function $v(x, y)$.

Let $v(x, y) = P(x)Q(y)$. Then $\Delta v = 0$ implies

$$P''(x) - \lambda P(x) = 0, \quad Q''(y) + \lambda Q = 0, \quad \lambda \in \mathbb{R}. \quad (7.110)$$

From the boundary conditions on the sides $x = 0$ and $x = 1$ we obtain

$$v(0, y) = P(0)Q(y) = 0, \quad v(1, y) = P(1)Q(y) = 0, \quad (7.111)$$

that is, $P(0) = P(1) = 0$. Therefore the function $P(x)$ satisfies the Sturm–Liouville problem

$$P''(x) - \lambda P(x) = 0, \quad P(0) = P(1) = 0, \quad (7.112)$$

whose solution is

$$\lambda_n = -(n\pi)^2, \quad P_n(x) = \sin(n\pi x), \quad n \geq 1. \quad (7.113)$$

It follows that the function $Q(y)$ satisfies the equation

$$Q''(y) - (n\pi)^2 Q(y) = 0. \quad (7.114)$$

The general solution of equation (7.114) can be written as a superposition of linearly independent solutions

$$Q_n(y) = A_n \sinh(n\pi y) + B_n \sinh(n\pi(y - 1)), \quad n \geq 1. \quad (7.115)$$

Hence the solution of Laplace's equation is given in the form of the series

$$v(x, y) = \sum_{n=1}^{\infty} P_n(x)Q_n(y) \quad (7.116)$$

$$= \sum_{n=1}^{\infty} \left[A_n \sinh(n\pi y) + B_n \sinh(n\pi(y - 1)) \right] \sin(n\pi x). \quad (7.117)$$

From the boundary condition on the side $y = 0$ we obtain

$$v(x, 0) = \sum_{n=1}^{\infty} B_n \sinh(-n\pi) \sin(n\pi x) = \sin(\pi x), \quad (7.118)$$

which implies $B_1 = -1/\sinh(\pi)$ and $B_n = 0$ for $n \geq 2$. Similarly, on the side $y = 1$ we have

$$v(x, 1) = \sum_{n=1}^{\infty} A_n \sinh(n\pi) \sin(n\pi x) = 0, \quad (7.119)$$

so $A_n = 0$ for all $n \geq 1$. Therefore the function $v(x, y)$ has the simple form

$$v(x, y) = -\frac{1}{\sinh(\pi)} \sinh(\pi(y-1)) \sin(\pi x). \quad (7.120)$$

Finally, the solution of the original problem is given by

$$u(x, y) = v(x, y) + P_2(x, y) = 1 + y - \frac{1}{\sinh(\pi)} \sinh(\pi(y-1)) \sin(\pi x). \quad (7.121)$$

□

7.2.2 Circular Domains

Another example of a domain on which Laplace's equation can be solved by separation of variables is the disk. Let $K_a \subset \mathbb{R}^2$ be the disk of radius $a > 0$ centered at the origin. Consider the Dirichlet problem

$$\Delta u(x, y) = 0, \quad (x, y) \in K_a, \quad (7.122)$$

$$u(x, y) = g(x, y), \quad (x, y) \in \partial K_a, \quad (7.123)$$

where g is a continuous function on ∂K_a . Because of the symmetry of the domain, introduce polar coordinates $x = r \cos(\theta)$, $y = r \sin(\theta)$ and define the function $w(r, \theta) = u(r \cos(\theta), r \sin(\theta))$. Laplace's equation in polar coordinates can be

obtained as follows. The derivatives of the function $w(r, \theta)$ are

$$\frac{\partial w}{\partial r} = u_x \cos(\theta) + u_y \sin(\theta), \quad (7.124)$$

$$\frac{\partial^2 w}{\partial r^2} = u_{xx} \cos^2(\theta) + 2u_{xy} \sin(\theta) \cos(\theta) + u_{yy} \sin^2(\theta), \quad (7.125)$$

$$\frac{\partial w}{\partial \theta} = -u_x r \sin(\theta) + u_y r \cos(\theta), \quad (7.126)$$

$$\begin{aligned} \frac{\partial^2 w}{\partial \theta^2} &= r^2(u_{xx} \sin^2(\theta) + u_{yy} \cos^2(\theta)) - 2u_{xy} r^2 \sin(\theta) \cos(\theta) \\ &\quad - r(u_x \cos(\theta) + u_y \sin(\theta)). \end{aligned} \quad (7.127)$$

From equations (7.125) and (7.127) it follows that

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = u_{xx} + u_{yy} - \frac{1}{r}(u_x \cos(\theta) + u_y \sin(\theta)). \quad (7.128)$$

Substituting equation (7.124) into (7.128), we obtain

$$u_{xx} + u_{yy} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2}. \quad (7.129)$$

Thus the Laplace operator in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \quad (7.130)$$

so the Dirichlet problem becomes

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} = 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 < r < a. \quad (7.131)$$

On the boundary of the disk, the function $w(r, \theta)$ satisfies

$$w(a, \theta) = g(a \cos(\theta), a \sin(\theta)), \quad 0 \leq \theta \leq 2\pi. \quad (7.132)$$

Since we are interested in continuous solutions on the disk K_a (which have no singularity at the origin), we must add the condition that $\lim_{r \rightarrow 0^+} w(r, \theta)$ be finite.

We seek the solution in the form $w(r, \theta) = P(r)Q(\theta)$. Substituting into equation (7.131) and separating variables, we obtain

$$\frac{r^2 P''(r) + r P'(r)}{P(r)} = \lambda, \quad \frac{Q''(\theta)}{Q(\theta)} = -\lambda, \quad \lambda \in \mathbb{R}. \quad (7.133)$$

It follows that the functions $P(r)$ and $Q(\theta)$ satisfy the ordinary differential equations

$$r^2 P''(r) + rP'(r) - \lambda P(r) = 0, \quad 0 < r < a, \quad (7.134)$$

$$Q''(\theta) + \lambda Q(\theta) = 0, \quad 0 \leq \theta \leq 2\pi. \quad (7.135)$$

The values $\theta = 0$ and $\theta = 2\pi$ represent the same point on the circle, so $Q(\theta)$ satisfies the periodic boundary condition $Q(0) = Q(2\pi)$. The Sturm–Liouville problem (7.135) has solutions for the eigenvalues $\lambda_n = n^2$, $n = 0, 1, 2, \dots$, with corresponding eigenfunctions

$$Q_0(\theta) = A_0, \quad n = 0, \quad (7.136)$$

$$Q_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta), \quad n = 1, 2, 3, \dots \quad (7.137)$$

This implies that the radial part of the solution satisfies the equation

$$r^2 P''(r) + rP'(r) - n^2 P(r) = 0. \quad (7.138)$$

For $n = 0$ the solution is obtained by direct integration:

$$P_0(r) = C_0 + D_0 \ln(r). \quad (7.139)$$

For $n > 0$ we seek the solution in the form $P(r) = r^k$, which yields the indicial equation $k^2 - n^2 = 0$, that is, $k = \pm n$. Hence the equation has the general solution

$$P_n(r) = C_n r^n + D_n r^{-n}, \quad n = 1, 2, 3, \dots \quad (7.140)$$

Since the desired solution has no singularity at $r = 0$, the coefficients D_n vanish, and we obtain

$$P_0(r) = C_0, \quad n = 0, \quad (7.141)$$

$$P_n(r) = C_n r^n, \quad n = 1, 2, 3, \dots \quad (7.142)$$

The general solution of Laplace's equation on the disk is therefore given by the superposition

$$w(r, \theta) = \sum_{n=0}^{\infty} P_n(r) Q_n(\theta) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} r^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)), \quad (7.143)$$

where $\alpha_0 = 2A_0C_0$, $\alpha_n = A_nC_n$, and $\beta_n = B_nC_n$ are coefficients determined by the boundary condition (7.132). Define the function $h(\theta) = g(a \cos(\theta), a \sin(\theta))$. Then from equations (7.132) and (7.143) it follows that

$$\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} a^n (\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)) = h(\theta), \quad 0 \leq \theta \leq 2\pi. \quad (7.144)$$

If h is continuous and piecewise C^1 on $[0, 2\pi]$, then from the condition $h(0) = h(2\pi)$ it follows that the Fourier series (7.144) converges uniformly to h . The Fourier coefficients are given by

$$\alpha_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \cos(n\varphi) d\varphi, \quad n = 0, 1, 2, \dots, \quad (7.145)$$

$$\beta_n = \frac{1}{\pi a^n} \int_0^{2\pi} h(\varphi) \sin(n\varphi) d\varphi, \quad n = 1, 2, 3, \dots \quad (7.146)$$

Thus the solution of the Dirichlet problem on the disk is completely determined.

7.2.3 Poisson's Formula

Relations (7.143) and (7.145)–(7.146) give the solution of Laplace's equation on the disk in the form of an infinite series. The same solution can be written in an integral representation known as Poisson's formula. In this representation the solution is given as an integral over the boundary of the domain of the function $h(\varphi)$ multiplied by the Poisson kernel. Substituting equations (7.145) and (7.146) into (7.143), we obtain

$$w(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\varphi) d\varphi + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_0^{2\pi} \left(\frac{r}{a}\right)^n h(\varphi) \cos(n(\theta - \varphi)) d\varphi, \quad (7.147)$$

where we used the identity

$$\cos(n(\theta - \varphi)) = \cos(n\theta) \cos(n\varphi) + \sin(n\theta) \sin(n\varphi).$$

Define the functions

$$f_n(\varphi) = \left(\frac{r}{a}\right)^n h(\varphi) \cos(n(\theta - \varphi)), \quad n \geq 1. \quad (7.148)$$

The function $h(\varphi)$ is continuous on the closed set $[0, 2\pi]$, so there exists $M > 0$ such that $|h(\varphi)| \leq M$ for every $\varphi \in [0, 2\pi]$. Hence the function $f_n(\varphi)$ is bounded by

$$\sup_{0 \leq \varphi \leq 2\pi} |f_n(\varphi)| \leq \left(\frac{r}{a}\right)^n M. \quad (7.149)$$

The series $\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n$ converges for every $0 \leq r < a$, so by the Weierstrass test and (7.149), the series $\sum_{n=1}^{\infty} f_n(\varphi)$ converges uniformly on $[0, 2\pi]$. Therefore we may interchange the integral and the summation in equation (7.147), which yields

$$w(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} h(\varphi) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\theta - \varphi)) \right] d\varphi. \quad (7.150)$$

Our next task is to determine the sum of the series

$$\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\theta - \varphi)). \quad (7.151)$$

Define $\rho = r/a$, $\alpha = \theta - \varphi$, and $z = \rho e^{i\alpha}$. By de Moivre's formula,

$$z^n = \rho^n (\cos(n\alpha) + i \sin(n\alpha)),$$

which implies

$$\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\theta - \varphi)) = \operatorname{Re} \left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n \right). \quad (7.152)$$

Since $|z| = r/a < 1$, the geometric series converges and

$$\sum_{n=1}^{\infty} z^n = \frac{z}{1-z}.$$

Therefore

$$\frac{1}{2} + \sum_{n=1}^{\infty} z^n = \frac{1+z}{2(1-z)} = \frac{1-\rho^2 + i 2\rho \sin(\alpha)}{2(1-2\rho \cos(\alpha) + \rho^2)}, \quad (7.153)$$

and taking the real part gives

$$\begin{aligned} \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n(\theta - \varphi)) &= \frac{1-\rho^2}{2(1-2\rho \cos(\alpha) + \rho^2)} \\ &= \frac{a^2 - r^2}{2(a^2 - 2ar \cos(\theta - \varphi) + r^2)}. \end{aligned} \quad (7.154)$$

Substituting equation (7.154) into (7.150), we obtain Poisson's formula

$$w(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \varphi) + r^2} h(\varphi) d\varphi. \quad (7.155)$$

Equation (7.155) gives the harmonic function $w(r, \theta)$ on the disk K_a in terms of its values on the boundary ∂K_a . The integrand

$$K(r, \varphi; a, \theta) = \frac{a^2 - r^2}{a^2 - 2ar \cos(\theta - \varphi) + r^2} \quad (7.156)$$

is called the Poisson kernel. Note that for $r = 0$ we recover the mean value theorem for harmonic functions, since at the origin

$$w(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\varphi) d\varphi = \frac{1}{2\pi a} \int_{\partial K_a} h ds. \quad (7.157)$$

The Poisson kernel is an example of a Green function by means of which the solution of a differential equation can be written in an integral representation that incorporates the boundary conditions. Green functions for Laplace's equation can be constructed for various boundary conditions and various domains, bounded as well as unbounded.