DIFFERENTIAL CALCULUS - EXERCISES

4.1 Functions

1. Suppose that t hours past midnight, the temperature in Rome was $C(t) = -\dfrac{1}{6}t^2 + 4t + 10$ degrees Celsius.

   (a) What was the temperature at 2:00 P.M.?

   (b) By how much the temperature increases or decreases between 6:00 and 9:00 PM?

   Solution.

   (a) We put $t = 14$ in C and we have
   $$C(14) = -\tfrac{1}{6} \cdot 14^2 + 4 \cdot 14 + 10 = -\tfrac{98}{3} + 66 = \tfrac{-98+198}{3} = \tfrac{100}{3},$$
   so the temperature was $33\frac{1}{3}C^0$.

   (b) Calculate $C(t)$ for $t = 18$ and $t = 21$:
   $$C(18) = -\tfrac{1}{6} \cdot 18^2 + 4 \cdot 18 + 10 = -54 + 82 = 28,$$
   $$C(21) = -\tfrac{1}{6} \cdot 21^2 + 4 \cdot 21 + 10 = -\tfrac{147}{2} + 94 = \tfrac{41}{2}.$$
   Since $28 > \frac{41}{2}$, we can conclude that temperature decreased for $28 - \frac{41}{2} = \frac{56-41}{2} = \frac{15}{2} = 7.5$ Celsius' degree.

2. Specify the domain and the range of the given function:

   (a) $h(u) = \dfrac{1}{\sqrt{u^2 - 9}}$,

   (b) $g(t) = \dfrac{1}{\mid t - 1 \mid}$,

   (c) $f(x) = \sqrt{\dfrac{x}{x+1}}$.

Solution.

(a) Since division by any real number except zero is possible and since negative numbers do not have real roots , the only values of u for which $h(u)$ can be evaluated are those for which $u^2 - 9 \geq 0$ and $u^2 - 9 \neq 0$. These condition imply $u^2 - 9 = (u-3)(u+3) > 0$. Now, we conclude that the domain of h are all real numbers u for which $\mid u \mid > 3$. The range of h is the set of all real numbers t for which $t > 0$

(b) Since $\mid t-1 \mid = 0$ implies $t = 1$ the domain of g are all real numbers t except $t = 1$. The range of g is the set of all real numbers x for which $x > 0$.

(c) The only values of x for which $f(x)$ can be evaluated are those for which $\frac{x}{x+1} \geq 0$ and $x + 1 \neq 0$. Let us sketch the following table.

Values of x	$x < -1$	-1	$-1 < x < 0$	0	$x > 0$
x	$-$	-1	$-$	0	$+$
$x+1$	$-$	0	$+$	1	$+$
$\dfrac{x}{x+1}$	$+$	undefined	$-$	0	$+$

Now, we conclude that the domain of f are all real numbers x for which $x < -1$ or $x \geq 0$. It remains to specify the range of f. Note that $f(x) \geq 0$ for each x in the domain of f. However $f(x) \neq 1$, since $f(x) = 1$ would imply $x = x + 1$, which is impossible. Thus, the range of f is the set of all real numbers t for which $t \geq 0$ and $t \neq 1$.

1. Let $f(x) = \sqrt{x}$ and $g(x) = -\dfrac{1}{x^2+1}$ be given. Examine if the compositions $g \circ f$ and $f \circ g$ are defined.

   Solution. Notice that the domain of f is the set of all real numbers x for which $x \geq 0$. Furthermore, the range of f is equal to its domain. On the other hand, the domain of g is the set of all real numbers x and the range of g is the set of all real numbers t for which $-1 \leq t < 0$. Since the range of f is contained in the domain of g, the composition $g \circ f$ is defined. Since the range of g is not contained in the domain of f, the composition $f \circ g$ is not defined.

4. Find the composite function $(g \circ f)(x)$ if

   (a) $f(x) = x^2 - 1, \quad g(u) = \sqrt{u+1},$

(b) $f(x) = \dfrac{x-1}{x+1}, \quad g(t) = t^3,$

(c) $f(x) = x^3, \quad g(x) = \dfrac{x-1}{x+1}.$

Solution.

(a) $(g \circ f)(x) = g(f(x)) = g(x^2-1) = \sqrt{(x^2-1)+1} =$
$= \sqrt{x^2} = \mid x \mid = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$

(b) $(g \circ f)(x) = g(f(x)) = g\left(\frac{x-1}{x+1}\right) = \left(\frac{x-1}{x+1}\right)^3$

(c) $(g \circ f)(x) = g(f(x)) = g\left(x^3\right) = \frac{x^3-1}{x^3+1}$

Notice that from (b) and (c) follows that, in general,

$$(f_2 \circ f_1)(x) \neq (f_1 \circ f_2)(x),$$

which means that for the composite function the commutativity is not valid.

5. Find the functions $g(x)$ and $h(u)$ such that $f(x) = (h \circ g)(x)$ if

(a) $f(x) = \sqrt{x+3} - \dfrac{1}{(x+4)^3},$

(b) $f(x) = \dfrac{x^2-1}{x^2+1}.$

Solution.

(a) Since $f(x) = \sqrt{x+3} - \frac{1}{(x+4)^3} = \sqrt{x+3} - \frac{1}{(x+3+1)^3}$ you can put $g(x) = x+3$ and $h(u) = \sqrt{u} - \frac{1}{(u+1)^3}$. Another possible choice is $g(x) = x+4$ and $h(u) = \sqrt{u-1} - \frac{1}{u^3}$.

(b) Since $f(x) = \frac{x^2-1}{x^2+1} = \frac{(x^2+1)-2}{x^2+1} = \frac{x^2-1}{(x^2-1)+2}$ you can choose
$g(x) = x^2$ and $h(u) = \frac{u-1}{u+1}$ or
$g(x) = x^2+1$ and $h(u) = \frac{u-2}{u}$ or
$g(x) = x^2-1$ and $h(u) = \frac{u}{u+2}$.

6. Examine if $g(x) = \frac{1}{\sqrt{x}}$ is the inverse function of $f(x) = \frac{1}{x^2}$.

Solution. The domain of f is the set of all real numbers x except $x = 0$. The range of f is the set of all real numbers t for which $t > 0$. The domain of g is equal to the range of f, and the codomain of g equals its domain. Therefore both compositions $g \circ f$ and $f \circ g$ are

defined. Let $x > 0$ be an arbitrary element in the domain of g and let us compute $(f \circ g)(x) = f(g(x)) = f(\frac{1}{\sqrt{x}}) = \frac{1}{\left(\frac{1}{\sqrt{x}}\right)^2} = x$. Now, we can see that $(f \circ g)(x) = x$, for each x in the domain of g. Let $x \neq 0$ be an arbitrary element in the domain of f.Let us compute $(g \circ f)(x) = g(f(x)) = g(\frac{1}{x^2}) = \frac{1}{\sqrt{\frac{1}{x^2}}} = \sqrt{x^2} =\mid x \mid$. Since for each negative number $\mid x \mid= -x \neq x$, we conclude that $(g \circ f)(x) = x$ is not valid for each x in the domain of f. Therefore, $g(x) = \frac{1}{\sqrt{x}}$ is not the inverse function of $f(x) = \frac{1}{x^2}$.

7. Examine if the function $f(x) = \sqrt{x^2 + 2}$ has the inverse function.

   Solution. Let $x \neq 0$ be an arbitrary real number. Obviously x is in the domain of f,since the domain of f is the set of all real numbers. Then $-x \neq x$ and $f(-x) = \sqrt{(-x)^2 + 2} = \sqrt{x^2 + 2} = f(x)$. According to Theorem 3.13 f has no inverse function.

8. Find the inverses of the following functions:

   (a) $f(x) = \dfrac{3}{x^3 + 1}$,

   (b) $f(x) = \dfrac{2x - 1}{x + 2}$.

   Solution.

   (a) First note that the domain of f is the set of all real numbers x except $x = -1$ and its range is the set of all real numbers t except $t = 0$. It is easy to check that the function f satisfies the condition (i) of Theorem 3.13. Hence, we conclude that $f : A \to B$, where $A = \mathbb{R} \setminus \{-1\} = \{x \in \mathbb{R} : x \neq -1\}$ and $B = \mathbb{R} \setminus \{0\} = \{x \in \mathbb{R} : x \neq 0\}$, admits the inverse function $g : B \to A$. Let us find the rule of assignment of g. First put $y = \frac{3}{x^3+1}$ and then express x in terms of y. You get

   $$y(x^3 + 1) = 3, \quad x^3 + 1 = \tfrac{3}{y}, \quad x = \sqrt[3]{\tfrac{3}{y} - 1}, \quad x = \sqrt[3]{\tfrac{3-y}{y}}.$$

   Interchange x and y in the last equation to get an expression for the function $y = \sqrt[3]{\frac{3-x}{x}}$. So, $g(x) = \sqrt[3]{\frac{3-x}{x}}$ is the inverse function of $f(x)$.

   (b) First note that the domain of f is the set of all real numbers x except $x = -2$ and its range is the set of all real numbers t except $t = 2$. It is easy to check that the function f satisfies the condition

(i) of Theorem 3.13. Hence, we conclude that $f : A \to B$, where $A = \mathbb{R} \setminus \{-2\}$ and $B = \mathbb{R} \setminus \{2\}$, admits the inverse function $g : B \to A$. Let us find the rule of assignment of g. From $y = \frac{2x-1}{x+2}$ you get

$$\begin{aligned} y(x+2) &= 2x-1, \quad yx - 2x = -2y - 1, \\ x(y-2) &= -(2y+1), \quad x = \tfrac{1+2y}{2-y}. \end{aligned}$$

After interchanging x and y you get $g(x) = \frac{1+2x}{2-x}$, which is the inverse function of $f(x)$.

9. Show that the function $f(x) = \dfrac{5x+3}{2x-5}$ coincides with its inverse.

Solution. You need to find the inverse function $g(x)$ of $f(x)$ and show $f(x) = g(x)$. Note that the domain of f is the set $A = \mathbb{R} \setminus \left\{\frac{5}{2}\right\}$ and the range of f is equal to its domain. It is easy to check that the function f satisfies the condition (i) of Theorem 3.13. Hence, we conclude that $f : A \to A$ has the inverse function $g : A \to A$. From $y = \frac{5x+3}{2x-5}$ you get

$$y(2x-5) = 5x+3, \quad yx - 5x = 3+5y, \quad x(2y-5) = 3+5y, \quad x = \tfrac{5y+3}{2y-5}.$$

Since $g(x) = \frac{5x+3}{2x-5} = f(x)$, it follows that the function $f(x) = \frac{5x+3}{2x-5}$ coincides with its inverse.

10. Write an equation for the line that passes through the points $(2,3)$ and $(4,5)$.

Solution. The general form of the linear equation is

$$y = ax + b, \quad a \neq \emptyset.$$

Recall that the coefficient a is the slope $\frac{y_2-y_1}{x_2-x_1}$ of the line and the coefficient b is the height at which the line crosses the y axis. Put $(2,3) = (x_1, y_1)$ and $(4,5) = (x_2, y_2)$ and compute

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{4-2} = 1.$$

Now, use the point-slope form of the equation of a line

$$y - y_1 = a(x - x_1)$$

to get $y - 3 = 1(x-2)$. The required equation is $y = x + 1$.

11. Find the slope of the line that passes through the points of intersection of the graphs of the functions $f(x) = x^3$ and $g(x) = x^3 + x^2 - 3x + 2$.

    Solution. To find the points of intersection algebraically, set $f(x)$ equal to $g(x)$ and solve for x. You get

    $$x^3 = x^3 + x^2 - 3x + 2 \quad x^2 - 3x + 2 = (x-2)(x-1) = 0.$$

    Thus, the points of intersection are $(2, 8)$ and $(1, 1)$. Since

    $$\text{Slope } = \frac{y_2 - y_1}{x_2 - x_1}$$

    you get Slope $= \frac{1-8}{1-2} = 7$.

12. Biologist have determined that under ideal conditions, the number of bacteria in a culture grows exponentially. Suppose that 2000 bacteria are initially present in a certain culture and that 6000 are present 20 minutes later. How many bacteria will be present at the end of 1 hour?

    Solution. Let $Q(t)$ denote the number of bacteria present after t minutes. Since the number of bacteria grows exponentially and since 2000 bacteria were initially present, you know that Q is a function of the form

    $$Q(t) = 2000e^{kt}.$$

    Since 6000 bacteria are present after 20 minutes, it follows

    $$6000 = 2000e^{20k} \qquad \text{or} \qquad e^{20k} = 3.$$

    To find the number of bacteria present at the end of 1 hour (60 minutes), compute $Q(60)$. You get

    $$Q(60) = 2000e^{60k} = 2000\left(e^{20k}\right)^3 = 2000\,(3)^3 = 54000.$$

    That is, 54000 bacteria will be present at the end of 1 hour.

13. The population density x miles from the center of a city is given by a function of the form

    $$Q(x) = Ae^{-kx.}.$$

    Find this function if it is known that the population density at the center of the city is 15000 people per square mile and the density 10 miles from the center is 9000 people per square mile.

Solution. For simplicity, express the density in units of 1000 people per square mile. The fact that $Q(0) = 15$ tells you that $Q(0) = A = 15$. The fact $Q(10) = 15e^{-10k} = 9$ tells you

$$e^{-10k} = \tfrac{9}{15} \qquad \text{or} \qquad e^{-10k} = \tfrac{3}{5}.$$

Taking the logarithm of each side of this equation, you get

$$-10k \ln e = \ln \tfrac{3}{5}, \qquad -10k = \ln \tfrac{3}{5}, \qquad k = -\tfrac{1}{10} \ln \tfrac{3}{5} \approx 0.051.$$

Hence the exponential function for the population density is (approximately) $Q(x) = 15e^{-0.051x}$.

Homework

1. It is estimated that t years from now, the population of a certain suburban community will be $P(t) = 20 - \dfrac{6}{t+1}$ thousand.

   (a) What will the population of the community be 9 years from now?

   (b) By how much will the population increase during the 9th year?

   (c) What will happen to the size of the population in the long run?

2. Biologists have found that the speed of blood in an artery is a function of the distance of the blood from the artery's central axis. According to **Poiseuille's law**, the speed (in centimeters per second) of blood that is r centimeters from the central axis of an artery is given by the function

$$S(r) = C(R^2 - r^2),$$

   where C is a constant and R is the radius of the artery. Suppose that for a certain artery, $C = 1.76 \times 10^5$ and $R = 1.2 \times 10^{-2}$ centimeters.

   (a) Compute the speed of the blood at the central axis of this artery.

   (b) Compute the speed of the blood midway between the artery's wall and central axis.

In Problems 3 through 6, specify the domain of the given function.

3. $f(x) = \dfrac{\sqrt{x^2 - 4}}{\sqrt{x - 4}}$

4. $h(u) = \dfrac{u}{u^2 - u - 2}$

5. $g(t) = \sqrt{|\, t - 3 \,|}$

6. $h(x) = \dfrac{1}{\sqrt[3]{x^2 - 1}}$

   In Problems 7 through 9, find the composite function $(g \circ f)(x)$.

7. $f(x) = \dfrac{x+1}{x}$, $g(u) = \dfrac{1}{u^2+1} - u$

8. $f(x) = \sqrt[3]{x-1}$, $g(t) = t^3 - 2t^2$

9. $f(x) = \sqrt{x-1}$, $g(x) = 2x^2 + 1$

   In Problems 10 through 12, find the functions $g(x)$ and $h(u)$ such that $f(x) = (h \circ g)(x)$.

10. $f(x) = \sqrt{x+2} - \dfrac{1}{(x+1)^2}$

11. $f(x) = \dfrac{x^3 - 1}{x^3 + 1}$

12. $f(x) = \dfrac{\sqrt{x+2}}{\sqrt[3]{x+2}}$

13. An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air will be $c(p) = 0.4p + 1$ parts per million when the population is p thousand. It is estimated that t years from now the population will be $p(t) = 8 + 0.2t^2$ thousand.

    (a) Express the level of carbon monoxide in the air as a function of time?

    (b) What will the carbon monoxide level be 2 years from now?

    (c) When will the carbon monoxide level reach 6.2 parts per million?

14. Find an equation of the line that passes through the points $(2, 6)$ and $(4, 10)$

15. Write an equation for the line that passes through the points of intersection of the graphs of the functions $f(x) = x^2$ and $g(x) = 2x^2 - 2x - 3$.

16. Examine if the function $g(x) = \log_2(x+1)$ is the inverse function of $f(x) = 2^x - 1$.

In Problems 17 through 20, find the inverse function $g(x)$ of the given function.

17. $f(x) = \dfrac{x+1}{x}$
18. $f(x) = \ln(x+2)$
19. $f(x) = x^5 + 9$
20. $f(x) = \sqrt[3]{x+1}$
21. A certain industrial machine depreciates so that its value after t years is given by a function of the form $Q(t) = Q_0 e^{-0.04t}$. After 20 years, the machine is worth 10000\$. What was its original value?
22. On the Richter scale, the magnitude of an earthquake of intensity I is given by
$$R = \frac{\ln I}{\ln 10}.$$
    (a) Find the intensity of the 1906 San Francisco earthquake, which measured 8.3 on the Richter scale.
    (b) How much more intense was the San Francisco earthquake of 1906 than the Bay Area Series earthquake of 1989, which measured 6.9 on the Richter scale.

Results

1. (a) $P(9) = 19.4$ thousand
   (b) $P(9) - P(8) = \frac{1}{15}$thousand
   (c) The size will reach 20000 thousand.
2. (a) $S(0) = CR^2 = 25.344$ (centimeters per second)
   (b) $S(\frac{R}{2}) = \frac{3}{4}CR^2 = 19.008$ (centimeters per second)
3. All real numbers x for which $x > 4$.
4. All real numbers u except $u = 2$ and $u = -1$.

5. All real numbers t.
6. All real numbers x except $x = 1$and $x = -1$.
7. $(g \circ f)(x) = \frac{x^2}{2x^2+2x+1} - \frac{x+1}{x}$
8. $(g \circ f)(x) = x - 1 - 2\sqrt[3]{x^2 - 2x + 1}$
9. $(g \circ f)(x) = 2 \mid x - 1 \mid +1$
10. One possible choice is $g(x) = x + 1$ and $h(u) = \sqrt{u+1} - \frac{1}{u^2}$.
11. One possible choice is $g(x) = x^3 + 1$ and $h(u) = \frac{u-2}{u}$.
12. One possible choice is $g(x) = x + 2$ and $h(u) = \frac{\sqrt{u}}{\sqrt[3]{u}}$.
13. (a) $c(p(t)) = 0.08t^2 + 4.2$
    (b) $c(p(2)) = 4.52$ parts per million
    (c) $c(p(t)) = 6.2$ implies $t = 5$ years from now.
14. $y = 2x + 2$
15. $y = 2x + 3$
16. Yes
17. $g(x) = \frac{1}{x-1}$
18. $g(x) = e^x - 2$
19. $g(x) = \sqrt[5]{x - 9}$
20. $g(x) = x^3 - 1$
21. $Q_0 = 10000e^{0.8}$
22. (a) $8.3 = \frac{\ln I}{\ln 10}$ implies $I = 10^{8.3}$
    (b) The San Francisco earthquake was $10^{8.3-6.9} = 10^{1.4}$ more intensive than The Bay Area Series earthquake.

4.2 Limits and Continuity

1. Examine if there exists the limit of the function $f(x) = \begin{cases} 2x+3, & x<2 \\ x-1, & x \geq 2 \end{cases}$ as x approaches 2.

   Solution. The function values $f(x)$ get closer and closer to 7 as x approaches 2 from the left. On the other hand, the function values $f(x)$ get closer and closer to 1 as x approaches 2 from the right. Hence, we conclude that the limit of the function $f(x) = \begin{cases} 2x+3, & x<2 \\ x-1, & x \geq 2 \end{cases}$ as x approaches 2 does not exist

2. Examine if there exists the limit of the function

$$f(x) = \begin{cases} x^2-1, & x<-1 \\ 2x^3+2, & x>-1 \end{cases}$$

   as x approaches to -1.

   Solution. Since the function values $f(x)$ get closer and closer to 0 as x approaches -1 from either side , it follows that $\lim\limits_{x\to -1} f(x) = 0$.

3. For what value of the constant A there exists the limit of the function $f(x) = \begin{cases} Ax-1, & x<3 \\ x^2-4, & x \geq 3 \end{cases}$ as x approaches 3.

   Solution. The function values $f(x)$ approach $3A-1$ as x approaches 3 from the left and approach 5 as x approaches 3 from the right. Thus, for $\lim\limits_{x\to 3} f(x)$ to exist, you must have $3A-1=5$, which means that $\lim\limits_{x\to 3} f(x)$ exists only for $A=2$.

4. Find the indicated limit if it exists.

   (a) $\lim\limits_{x\to 1} \left[x^2(7x+3)+2\right]$

   (b) $\lim\limits_{x\to 0} \dfrac{x^2-1}{x+1}$

   (c) $\lim\limits_{x\to -1} \dfrac{x^4-1}{x^2-1}$

   (d) $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4}$

   (e) $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

(f) $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

(g) $\lim\limits_{x\to 3} \dfrac{2x+3}{x-3}$

(h) $\lim\limits_{x\to 1} \dfrac{x^2+4x-5}{x^2-1}$

(i) $\lim\limits_{x\to 0} \dfrac{x^2+1}{x^2}$

(j) $\lim\limits_{x\to +\infty} \dfrac{2x^2-x+4}{x^2-4}$

(k) $\lim\limits_{x\to -\infty} \dfrac{1}{x^4+1}$

(l) $\lim\limits_{x\to +\infty} \dfrac{x^2+1}{x-1}$

Solution.

(a) $\lim\limits_{x\to 1} \left[x^2(7x+3)+2\right] = \lim\limits_{x\to 1} x^2 \cdot \lim\limits_{x\to 1}(7x+3) + \lim\limits_{x\to 1} 2 = 12$

(b) $\lim\limits_{x\to 0} \frac{x^2-1}{x+1} = \frac{\lim\limits_{x\to 0}(x^2-1)}{\lim\limits_{x\to 0}(x+1)} = \frac{-1}{1} = -1.$

(c) $\lim\limits_{x\to -1} \frac{x^4-1}{x^2-1} = \lim\limits_{x\to -1} \frac{(x^2-1)(x^2+1)}{x^2-1} = \lim\limits_{x\to -1}(x^2+1) = 2.$

(d) $\lim\limits_{x\to 2} \frac{x^3-8}{x^2-4} = \lim\limits_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)} = \lim\limits_{x\to 2} \frac{x^2+2x+4}{x+2} = \frac{12}{4} = 3.$

(e) $\lim\limits_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim\limits_{x\to 4} \frac{\sqrt{x}-2}{x-4}\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim\limits_{x\to 4} \frac{x-4}{(x-4)\sqrt{x}+2} = \lim\limits_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}.$

(f) $\lim\limits_{x\to 9} \frac{x-9}{\sqrt{x}-3} = \lim\limits_{x\to 9} \frac{x-9}{\sqrt{x}-3}\frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim\limits_{x\to 9} \frac{(x-9)\sqrt{x}+3}{x-9} = \lim\limits_{x\to 9}(\sqrt{x}+3) = 6.$

(g) Limit does not exist. Notice that $f(x)$ increases without any bound as x approaches 3 from the right and decreases without bound as x approaches 3 from the left.

(h) $\lim\limits_{x\to 1} \frac{x^2+4x-5}{x^2-1} = \lim\limits_{x\to 1} \frac{(x-1)(x+5)}{(x-1)(x+1)} = \lim\limits_{x\to 1} \frac{x+5}{x+1} = \frac{6}{2} = 3.$

(i) Limit does not exist, but since the values of $f(x)$ increase without bound as x gets closer to 0 from either side and do not approach any (finite) number we write $\lim\limits_{x\to 0} \frac{x^2+1}{x^2} = +\infty.$

(j) $\lim\limits_{x\to +\infty} \frac{2x^2-x+4}{x^2-4} = \frac{2}{1} = 2$

(k) $\lim\limits_{x\to -\infty} \frac{1}{x^4+1} = 0$

(l) $\lim\limits_{x\to +\infty} \frac{x^2+1}{x-1} = +\infty.$

5. Examine the convergence of the following sequences.

   (a) $a_n = (-1)^{n+1} \frac{1}{n^2+1}$

   (b) $a_n = (-1)^n \frac{n-1}{n+1}$

   (c) $a_n = \sqrt{n+1} - \sqrt{n}$

   Solution.

   (a) The sequence $((-1)^{n+1} \frac{1}{n^2+1})$ converges and $\lim\limits_{n\to\infty}[(-1)^{n+1} \frac{1}{n^2+1}] = 0$.

   (b) The sequence $\left((-1)^n \frac{n-1}{n+1}\right)$ diverges. Namely, a_{2n} approach 1 as n increases, but a_{2n+1} approach -1 as n increases. Therefore the limit does not exist.

   (c) The sequence $\left(\sqrt{n+1} - \sqrt{n}\right)$converges and
   $\lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) = \lim\limits_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} =$
   $\lim\limits_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty} \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$

6. The sum of the first and the fifth term of an arithmetic progression is equal to 26 and the product of the second by the fourth is 160. Find the sum of the first six terms of the progression.

   Solution. Since the nth term of the arithmetic progression is given by

   $$a_n = a + (n-1)d$$

   we get

   $$a_1 + a_5 = a + a + 4d = 26 \text{ and}$$
   $$a_2 a_4 = (a+d)(a+3d) = a^2 + 4ad + 3d^2 = 160.$$

   From the first equation it follows $a = 13 - 2d$, and thus

   $$(13-2d)^2 + 4(13-2d)d + 3d^2 = -d^2 + 169 = 160.$$

   It is obvious that two arithmetic progressions satisfy the required conditions. The first one is defined by $a = 7$ and $d = 3$, and he second one is defined by $a = 19$ and $d = -3$. Hence, we have two sums S_6^1 and S_6^2,

   $$S_6^1 = 3(a_1 + a_6) = 3(2a + 5d) = 3(14 + 15) = 87 \text{ and}$$
   $$S_6^2 = 3(a_1 + a_6) = 3(2a + 5d) = 3(38 - 15) = 69.$$

7. Find the sum of the first k natural numbers.

   Solution. The sequence $(n) = (1, 2, 3, \ldots, n, \ldots)$ of natural numbers is an instance of an arithmetic progression with difference $d = 1$. Since $a = 1$ you get
$$S_k = \frac{k(a_1 + a_k)}{2} = \frac{k(1+k)}{2}.$$

8. Find four numbers that constitute an alternating geometric progression in which the second term is less than the first by 35, and the third exceeds the fourth by 560.

   Solution. Since the nth term of the geometric progression is given by
$$a_n = aq^{n-1}$$
   we get $a_2 = a_1 - 35$ and $a_3 = a_4 + 560$. Now, we have two equations $a\,(q-1) = -35$ and $aq^2(1-q) = 560$. Since $a(1-q) = 35$ and $a(1-q) = \frac{560}{q^2}$ it follows $35q^2 = 560$. Thus $q = -4$ and $a = 7$. The required four numbers are
$$7, \ -28, \ 112, \ -448.$$

9. Find the sum of the first 5 terms of a geometric progression if the sum of its first three terms is $\frac{3}{4}$ and the second term is $-\frac{1}{2}$.

   Solution. Since $a_2 = aq = -\frac{1}{2}$ and $S_3 = a\frac{q^3-1}{q-1} = \frac{3}{4}$ you get $-\frac{1}{2q}\frac{q^3-1}{q-1} = -\frac{(q-1)(q^2+q+1)}{2q(q-1)} = \frac{3}{4}$ or $4(q^2+q+1)+6q = 0$. The solutions of that equation are $q = -2$ and $q = -\frac{1}{2}$. Thus we get two geometric progressions, the first one is defined by $a = \frac{1}{4}$ and $q = -2$ and the second one by $a = 1$ and $q = -\frac{1}{2}$. Thus we have two sums S_5^1 and S_5^2,
$$S_5^1 = a\frac{q^5-1}{q-1} = \frac{1}{4}\frac{-32-1}{-2-1} = \frac{11}{4},$$
$$S_5^2 = a\frac{q^5-1}{q-1} = \frac{-\frac{1}{32}-1}{-\frac{1}{2}-1} = \frac{\frac{33}{32}}{\frac{3}{2}} = \frac{11}{16}$$

10. Find A such that the function defined by $f(x) = \begin{cases} 2x+3, & x < 2 \\ Ax-1, & x \geq 2 \end{cases}$ be continuous for all x.

    Solution. Since $2x+3$ and $Ax-1$ are both linear functions, it follows that $f(x)$ will be continuous everywhere except possibly at $x = 2$. In order to f be continuous at $x = 2$, it has to be $\lim\limits_{x\to 2} f(x) = f(2) = 2A-1$. Since the values $f(x)$ approach 7 as x approaches 2 from the left and approach $2A-1$ as x approaches 2 from the right it has to be

    $2A - 1 = 7$ or $A = 4$.

11. Find A and B such that the function defined by

$$f(x) = \begin{cases} Ax^2 + 5x - 9, & x < 1 \\ B, & x = 1 \\ (3-x)(A-2x), & x > 1 \end{cases}$$

is continuous for all x.

Solution. Notice that $Ax^2 + 5x - 9$ and $(3-x)(A-2x)$ are both polynomials. Since the polynomials are continuous functions, the function f is continuous everywhere except possibly at $x = 1$. In order to f be continuous at $x = 1$, it has to be $\lim_{x\to 1} f(x) = f(1) = B$. Notice that the values $f(x)$ approach $A - 4$ as x approaches 1 from the left and approach $2A - 4$ as x approach 1 from the right. Hence, it has to be $A - 4 = B = 2A - 4$. Now, we get $A = 0$ and $B = -4$.

Homework

1. Examine if there exists the limit of $f(x) = \begin{cases} x+1, & x < 1 \\ 2-x, & x > 1 \end{cases}$ as x approaches 1.

In Problems 2 through 11, find the indicated limit if it exists.

2. $\lim_{x\to 3} \left[(x-1)^2 (x+1) - 1\right]$

3. $\lim_{x\to 0} \dfrac{x^2-1}{x^2+2}$

4. $\lim_{x\to 5} \dfrac{x+2}{x-5}$

5. $\lim_{x\to -2} \dfrac{x^2-x-6}{x^2+3x+2}$

6. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

7. $\lim_{x\to 0} \dfrac{1-x^2}{x^2}$

8. $\lim_{x\to +\infty} \dfrac{4x^3-x+2}{2x^3-x^2+2x}$

9. $\lim_{x\to -\infty} e^{3x}$

10. $\lim\limits_{x\to+\infty} \dfrac{x^3+1}{-x-1}$

11. $\lim\limits_{x\to 0} \ln x$

In Problems 12 through 14, examine the convergence of the given sequence.

12. $a_n = (-1)^n \dfrac{2n}{n+1}$

13. $a_n = \sqrt{\dfrac{4n+2}{4n}}$

14. $a_n = \dfrac{1}{3^n} - \dfrac{1}{\sqrt{n}}$

15. Find the sum of the first 100 even numbers (starting with 2).

16. Let (a_n) be a geometric progression defined by a and q, such that $\mid q \mid < 1$. Find $\lim\limits_{n\to\infty} S_n$, where S_n is the sum of the first n terms of the progression.

    In Problems 17 through 19, decide if the given function is continuous at the specified value of x.

17. $f(x) = \begin{cases} x+1, & x<1 \\ 3, & x=1 \\ x^2+x, & x>1 \end{cases}, \quad x=1$

18. $f(x) = \begin{cases} x-1, & x\le 2 \\ 2, & x>2 \end{cases}, \quad x=2$

19. $f(x) = \begin{cases} \dfrac{x^2-1}{x+1}, & x<-1 \\ x^2-3, & x\ge -1 \end{cases}, \quad x=-1$

20. Find A such that the function defined by $f(x) = \begin{cases} 3x+11, & x<1 \\ Ax-1, & x\ge 1 \end{cases}$ is continuous for all x.

Results

1. Limit does not exist.

2. 15

3. $-\frac{1}{2}$
4. Limit does not exist.
5. 5
6. 2
7. Limit does not exist $\left(\lim\limits_{x\to 0} \frac{1-x^2}{x^2} = +\infty\right)$.
8. 2
9. 0
10. Limit does not exist $\left(\lim\limits_{x\to +\infty} \frac{x^3+1}{-x-1} = -\infty\right)$.
11. Limit does not exist $\left(\lim\limits_{x\to 0} \ln x = -\infty\right)$.
12. (a_n) diverges.
13. (a_n) converges to 1.
14. (a_n) converges to 0.
15. $S_{100} = 10100$
16. $\lim\limits_{n\to\infty} S_n = \frac{a}{1-q}$
17. No
18. No
19. Yes
20. $A = 15$

4.3 The Derivative

1. Find the ratio of the increment of the function $f(x) = x^2 + 2$ and its argument when $x = 1$, if

   (a) $\Delta x = 1$

   (b) $\Delta x = \frac{1}{2}$

   (c) $\Delta x = \frac{1}{10}$

   (d) $\Delta x = \frac{1}{100}$

   Solution. We have to evaluate
   $$\frac{\Delta f(1)}{\Delta x} = \frac{f(1+\Delta x)-f(1)}{\Delta x} = \frac{(1+\Delta x)^2+2-(1^2+2)}{\Delta x} = \frac{(\Delta x)^2+2\Delta x}{\Delta x}.$$

   (a) $\frac{\Delta f(1)}{\Delta x} = \frac{(\Delta x)^2+2\Delta x}{\Delta x} = \frac{1+2}{1} = 3$

   (b) $\frac{\Delta f(1)}{\Delta x} = \frac{(\Delta x)^2+2\Delta x}{\Delta x} = \frac{\frac{1}{4}+1}{\frac{1}{2}} = \frac{5}{2}$

   (c) $\frac{\Delta f(1)}{\Delta x} = \frac{(\Delta x)^2+2\Delta x}{\Delta x} = \frac{\frac{1}{100}+\frac{2}{10}}{\frac{1}{10}} = \frac{21}{10}$

   (d) $\frac{\Delta f(1)}{\Delta x} = \frac{(\Delta x)^2+2\Delta x}{\Delta x} = \frac{\frac{1}{10000}+\frac{2}{100}}{\frac{1}{100}} = \frac{201}{100}$

2. Find the derivative of the function $f(x) = x^2 + 2$ when $x = 1$.

   Solution. We need to find $f'(1) = \lim\limits_{\Delta x \to 0} \frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{\Delta x^2+2\Delta x}{\Delta x} = \lim\limits_{\Delta x \to 0} (\Delta x + 2) = 2$.

3. Examine if the function $f(x) = \begin{cases} x^2 - 3, & x < -1 \\ 2x, & x \geq -1 \end{cases}$ has a derivative when $x = -1$.

   Solution. We need to examine if the limit
   $$f'(-1) = \lim_{\Delta x \to 0} \frac{f(-1+\Delta x)-f(-1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(-1+\Delta x)-(-2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(-1+\Delta x)+2}{\Delta x}$$
   exists. The values
   $$\frac{f(-1+\Delta x)+2}{\Delta x} = \frac{(-1+\Delta x)^2-3+2}{\Delta x} = \frac{(\Delta x)^2-2\Delta x}{\Delta x} = \Delta x - 2$$
   approach -2 as Δx approaches 0 from the left. On the other hand, the values
   $$\frac{f(-1+\Delta x)+2}{\Delta x} = \frac{2(-1+\Delta x)+2}{\Delta x} = \frac{2\Delta x}{\Delta x} = 2$$

are all equal 2 as Δx approaches 0 from the right. Hence, the limit $\lim\limits_{\Delta x \to 0} \frac{f(-1+\Delta x)-f(-1)}{\Delta x}$ does not exist, which implies that the function $f(x) = \begin{cases} x^2 - 3, & x < -1 \\ 2x, & x \geq -1 \end{cases}$ does not have a derivative when $x = -1$. Notice that the function f is continuous at $x = -1$.

4. Find the equation of the line that is tangent to the graph of the function $f(x) = 2x^2 + 1$ when $x = 2$.

   Solution. Since $f(2) = 9$, it follows that the point of tangency is $(2, 9)$. To find the slope of the tangent when $x = 2$, compute

$$f'(2) = \lim_{\Delta x \to 0} \frac{f(2+\Delta x)-f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2(2+\Delta x)^2+1-9}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x^2+8\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (2\Delta x + 8) = 8.$$

   Now use the point-slope formula $y - y_0 = a(x - x_0)$ with $a = 8$ and $(x_0, y_0) = (2, 9)$ to conclude that the equation of the tangent is $y - 9 = 8(x - 2)$ or $y = 8x - 9$.

5. Examine if the function $f(x) = \begin{cases} x^2, & x < 0 \\ x, & x \geq 0 \end{cases}$ is differentiable.

   Solution. We need to examine if the function f has a derivative when x is an arbitrary element in its domain. Therefore we distinguish three cases: $x < 0$, $x > 0$ and $x = 0$.

   (a) $x < 0$. Let us examine if the limit $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists. Since

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x,$$

   it follows that the function f is differentiable at each point $x < 0$.

   (b) $x > 0$. Let us examine if the limit $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists. Since

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x+\Delta x-x}{\Delta x} = 1,$$

   it follows that the function f is differentiable at each point $x > 0$.

   (c) $x = 0$. Let us examine if the limit $f'(0) = \lim\limits_{\Delta x \to 0} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$ exists. The values $\frac{f(\Delta x)}{\Delta x} = \frac{(\Delta x)^2}{\Delta x} = \Delta x$ approaches 0 as Δx approaches 0 from the left. On the other hand the values

$\frac{f(\Delta x)}{\Delta x} = \frac{\Delta x}{\Delta x} = 1$ are all equal 1 as Δx approaches 0 from the right. Now, we conclude that the limit $\lim\limits_{\Delta x \to 0} \frac{f(0+\Delta x)-f(0)}{\Delta x}$ does not exist and thus the function f is not differentiable. Note that f is continuous.

6. Examine if the function $f(x) = x^3$ is continuously differentiable.

   Solution. First we need to examine if the given function is differentiable at an arbitrary point in its domain. So, let us examine if the limit $f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists. Since

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\triangle x)^2 + (\triangle x)^3}{\Delta x} = \lim_{\Delta x \to 0} [3x^2 + 3x\triangle x + (\triangle x)^2] = 3x^2,$$

   we conclude that the function f is differentiable and its derivative is the function $f'(x) = 3x^2$, which is continuous. Thus, $f(x) = x^3$ is continuously differentiable function.

Homework

1. Find the ratio of the increment of the function $f(x) = 2x^2 + 3$ and its argument when $x = 0$, if

   (a) $\Delta x = \frac{1}{4}$

   (b) $\Delta x = \frac{1}{100}$

2. Find the derivative of the function $f(x) = 2x^2 + x$ when $x = -2$.

3. Examine if the function $f(x) = \begin{cases} x^2 - 4, & x \leq 3 \\ x + 2, & x > 3 \end{cases}$ has a derivative when $x = 3$.

4. Find the slope of the line that is tangent to the graph of the function $f(x) = 4x^2 + x + 11$ when $x = 2$.

In Problems 5 through 7 examine if the given function is differentiable.

5. $f(x) = x^2 + 2x + 6$

6. $f(x) = x^3 + 1$

7. $f(x) = \begin{cases} 2x+3, & x \leq 1 \\ 3x+2. & x > 1 \end{cases}$

Results

1. (a) $\frac{1}{2}$
   (b) $\frac{1}{50}$
2. $f'(-2) = -7$
3. It does not have
4. 17
5. Yes
6. Yes
7. No

4.4 Techniques of Differentiation

1. Differentiate the function $f(x) = -\dfrac{x^2}{16} + \dfrac{2}{x} - x^{3/2} + \dfrac{1}{3x^2} + \dfrac{x}{3}$.

   Solution. Applying the sum rule and the constant multiple rule you get

   $$f'(x) = -\tfrac{1}{16}\tfrac{d}{dx}(x^2) + 2\tfrac{d}{dx}(x^{-1}) - \tfrac{d}{dx}\left(x^{3/2}\right) + \tfrac{1}{3}\tfrac{d}{dx}(x^{-2}) + \tfrac{1}{3}\tfrac{d}{dx}(x).$$

   Now, use the power rule

   $$f'(x) = -\tfrac{1}{16}(2x) + 2(-x^{-2}) - \tfrac{3}{2}x^{1/2} + \tfrac{1}{3}(-2)x^{-3} + \tfrac{1}{3}$$

   and, at the end, simplify the result

   $$f'(x) = -\tfrac{1}{8}x - 2x^{-2} - \tfrac{3}{2}x^{1/2} - \tfrac{2}{3}x^{-3} + \tfrac{1}{3} = -\tfrac{1}{8}x - \tfrac{2}{x^2} - \tfrac{3}{2}\sqrt{x} - \tfrac{2}{3x^3} + \tfrac{1}{3}.$$

2. Differentiate the function $f(x) = 10(3x+1)(1-5x)$.

   Solution. According to the constant multiple rule and the product rule you get

   $$f'(x) = 10\left[(1-5x)\tfrac{d}{dx}(3x+1) + (3x+1)\tfrac{d}{dx}(1-5x)\right].$$

   Now apply the sum rule and, at the end simplify the result. You get $f'(x) = 10[3(1-5x) + (3x+1)(-5)] = -300x - 20$.

3. Differentiate the function $f(x) = \frac{x^2-3x+2}{2x^2+5x-1}$.

   Solution. Applying the quotient rule and the sum rule you get

$$f'(x) = \frac{d}{dx}\left(\frac{x^2-3x+2}{2x^2+5x-1}\right) = \frac{\left(2x^2+5x-1\right)\frac{d}{dx}\left(x^2-3x+2\right)-\left(x^2-3x+2\right)\frac{d}{dx}\left(2x^2+5x-1\right)}{\left(2x^2+5x-1\right)^2} = \frac{\left(2x^2+5x-1\right)(2x-3)-\left(x^2-3x+2\right)(4x+5)}{\left(2x^2+5x-1\right)^2} = \frac{11x^2-10x-7}{\left(2x^2+5x-1\right)^2}.$$

4. Find the equation of the line that is tangent to the graph of the function $f(x) = \dfrac{x+1}{x-1}$ at the point $T = (0,-1)$.

   Solution. First, let us find

$$f'(x) = \left(\frac{x+1}{x-1}\right)' = \frac{(x+1)'\cdot(x-1)-(x-1)'\cdot(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}.$$

   Then the slope of the required tangent is $a = f'(0) = -2$. Now, use the point-slope formula $y - y_0 = a(x - x_0)$ with $a = -2$ and $(x_0, y_0) = (0,-1)$ to conclude that the equation of the tangent is $y - (-1) = -2(x-0)$ or $y = -2x - 1$.

5. Find numbers a, b and c such that the graph of the function $f(x) = ax^2 + bx + c$ will have x intercepts at $(0,0)$ and $(5,0)$, and a tangent with slope 1 when $x = 2$.

   Solution. Let us find $f'(x) = (ax^2+bx+c)' = 2ax + b$. Since $f'(2) = 1$ we get the first equation $4a + b = 1$. Putting $x = 0$ and $y = 0$ in $y = ax^2 + bx + c$ we get $c = 0$. Similarly, putting $x = 5$ and $y = 0$ in $y = ax^2 + bx + c$ we get the third equation $25a + 5b + c = 0$. So, we have three simple equations with three unknowns

$$\begin{aligned} 4a + b &= 1 \\ c &= 0 \\ 25a + 5b + c &= 0 \end{aligned}$$

   The solution of this system is $a = -1$, $b = 5$ and $c = 0$. Thus, the desired function is $f(x) = -x^2 + 5x$.

6. Differentiate the function $f(x) = (2x+1)^4$.

   Solution. Notice that $f = h \circ g$ where $g(x) = 2x + 1$ and $h(x) = x^4$. Thus applying the chain rule we get $f'(x) = h'(g(x))\,g'(x) = 4(g(x))^3(2) = 8(2x+1)^3$. Usually, when we apply the chain rule, we do not write explicitly composite functions, as we did here, but we do

it tacitly in our head. Therefore, ordinary we apply the chain rule in the following way:

$$f'(x) = \left[(2x+1)^4\right]' = 4(2x+1)^3(2x+1)' = 4(2x+1)^3(2) = 8(2x+1)^3.$$

7. Differentiate the function $f(x) = \dfrac{1}{\sqrt{4x^2+1}}$.

Solution. Applying the chain rule we get

$$f'(x) = \left(\tfrac{1}{\sqrt{4x^2+1}}\right)' = \left((4x^2+1)^{-\frac{1}{2}}\right)' = -\tfrac{1}{2}(4x^2+1)^{-\frac{3}{2}}(4x^2+1)' = -\tfrac{1}{2}(4x^2+1)^{-\frac{3}{2}}8x = -4x(4x^2+1)^{-\frac{3}{2}} = -\tfrac{4x}{(4x^2+1)\sqrt{4x^2+1}}.$$

8. Differentiate the function $f(x) = \sqrt{\dfrac{3x+1}{2x-1}}$.

Solution. $f'(x) = \left(\sqrt{\frac{3x+1}{2x-1}}\right)' = \frac{1}{2}\frac{1}{\sqrt{\frac{3x+1}{2x-1}}}\left(\frac{3x+1}{2x-1}\right)' =$

$$\tfrac{1}{2}\sqrt{\tfrac{2x-1}{3x+1}}\tfrac{(3x+1)'\cdot(2x-1)-(2x-1)'(3x+1)}{(2x-1)^2} = \tfrac{1}{2}\sqrt{\tfrac{2x-1}{3x+1}}\tfrac{3(2x-1)-2(3x+1)}{(2x-1)^2} =$$

$$\tfrac{1}{2}\sqrt{\tfrac{2x-1}{3x+1}}\tfrac{-5}{(2x-1)^2} = -\tfrac{5}{2}\tfrac{1}{(2x-1)\sqrt{(2x-1)(3x+1)}}.$$

9. Differentiate the function $f(x) = (x+1)\ln\sqrt[3]{x^2+x}$.

Solution. Combine the product rule and the chain rule with the formula for the derivative of $\ln x$ to get

$$f'(x) = (x+1)'\ln\left(\sqrt[3]{x^2+x}\right) + (x+1)(\ln\sqrt[3]{x^2+x})' =$$

$$\ln\left(\sqrt[3]{x^2+x}\right) + (x+1)\tfrac{1}{\sqrt[3]{x^2+x}}\left(\sqrt[3]{x^2+x}\right)' =$$

$$\ln\left(\sqrt[3]{x^2+x}\right) + \tfrac{x+1}{\sqrt[3]{x^2+x}}\tfrac{1}{3}(x^2+x)^{-\frac{2}{3}}(2x+1) = \tfrac{(x+1)(2x+1)}{3(x^2+x)} + \ln\sqrt[3]{x^2+x}.$$

10. Differentiate the function $f(x) = xe^{\frac{x}{x+1}}$.

Solution. Combine the product rule, the quotient rule and the chain rule with the formula for the derivative of e^x to get

$$f'(x) = (x)'e^{\frac{x}{x+1}} + x\left(e^{\frac{x}{x+1}}\right)' = e^{\frac{x}{x+1}} + xe^{\frac{x}{x+1}}\tfrac{x+1-x}{(x+1)^2} =$$

$$\left(1 + \tfrac{x}{(x+1)^2}\right)e^{\frac{x}{x+1}} = \tfrac{x^2+3x+1}{(x+1)^2}e^{\frac{x}{x+1}}.$$

Homework

In Problems 1 through 7 find the derivative of the given function.

1. $f(x) = \sqrt{x^3} + \dfrac{1}{\sqrt{x^3}}$
2. $f(x) = (2x^2 - 4)(1 - 3x)$
3. $f(x) = \frac{x^2+1}{1-x^2}$
4. $f(x) = (x^5 - 5x^3 - 9)^7$
5. $f(x) = \sqrt{\dfrac{3x+2}{4x-1}}$
6. $f(x) = \dfrac{x+1}{x-1} \ln \dfrac{x-1}{x+1}$
7. $f(x) = \sqrt{xe^{3x-1}}$
8. Find the equation of the line that is tangent to the graph of the function $f(x) = x^5 - 3x^3 - 5x + 2$ at the point $(1, -5)$.
9. Find numbers a, b and c such that the graph of the function $f(x) = ax^2 + bx + c$ contains points $(0, 3)$ and $(5, 3)$, and a tangent with slope 1 when $x = 2$.

Results

1. $f'(x) = \frac{3}{2}\sqrt{x} - \frac{3}{2\sqrt{x^5}}$
2. $f'(x) = -18x^2 + 4x + 12$
3. $f'(x) = \frac{4x}{(x^2-1)^2}$
4. $f'(x) = 35x^2(x^2 - 3)(x^5 - 5x^3 - 9)^6$
5. $f'(x) = -\frac{11}{2}\frac{1}{(4x-1)\sqrt{(3x+2)(4x-1)}}$
6. $f'(x) = \frac{2}{(x-1)^2}\left(1 - \ln\frac{x-1}{x+1}\right)$
7. $f'(x) = \frac{1}{2}\frac{3x+1}{\sqrt{xe^{3x-1}}}e^{3x-1}$
8. $y = -9x + 4$
9. $a = -1,\ b = 5,\ c = 3.$

4.5 The Derivative as a Rate of Change

1. The gross national product (GNP)of a certain country was $N(t) = t^2 + 5t + 106$ billion dollars t years after 1980.

   (a) At what rate was the GNP changing with respect to time in 1988?

   (b) At what percentage rate was the GNP changing with respect to time in 1988?

   Solution.

   (a) The rate of change of the GNP is the derivative $N'(t) = 2t + 5$. The rate of change in 1988 was $N'(8) = 16+5 = 21$ billion dollars per year.

   (b) The percentage rate of change of the GNP in 1988 was

   $$\text{Percentage rate of change} = \frac{N'(8)}{N(8)}100 = \frac{21}{8^2 + 40 + 106} \cdot 100 = 10$$

   percent per year.

2. An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have assembled $f(x) = -x^3 + 6x^2 + 15x$ transistor radios x hours later.

   (a) Derive a formula for the rate at which the worker will be assembling radios after x hours.

   (b) At what rate will the worker be assembling radios at 9:00 A.M.?

   (c) How many radios will the worker assemble between 9:00 and 10:00 A.M.?

   Solution.

   (a) The rate of change is the derivative $f'(x) = -3x^2 + 12x + 15$.

   (b) $f'(1) = -3 + 12 + 15 = 24$ radios per hour.

   (c) $f(2) - f(1) = (-8 + 24 + 30) - (-1 + 6 + 15) = 26$ radios.

Homework

1. Find the percentage rate of change of the function $f(t) = 3t^2 - 7t + 5$ with the respect to t, when $t = 2$.

2. It is estimated that t years from now, the circulation of a local newspaper will be $C(t) = 100t^2 + 400t + 5000$.

   (a) Derive an expression for the rate at which the circulation will be changing with respect to time t years from now.

   (b) At what rate will the circulation be changing with respect to time 5 years from now? Will the circulation be increasing or decreasing at that time?

   (c) By how much will the circulation actually change during the 6th year?

Results

1. 166.67%.

2. (a) $C'(t) = 200t + 400$

   (b) Increasing at the rate of 1400 per year

   (c) 1500

4.6 Approximation by Differentials

1. An environmental study of a certain community suggests that t years from now the average label of carbon monoxide in the air will be $f(t) = 0.05t^2 + 0.1t + 3.4$ parts per million. By approximately how much will the carbon monoxide level change during the incoming six months.

   Solution. Let us find $f'(x) = \frac{5}{100}2t + \frac{1}{10} = \frac{1}{10}t + \frac{1}{10}$. Now we have to calculate

   $$\triangle f \approx \Delta t \cdot f'(0) = \frac{1}{2}\left(\frac{1}{10} \cdot 0 + \frac{1}{10}\right) = \frac{1}{20} = 0.05$$

   It will change approximately (remember that this is not an exact value, but only a fairly good approximation) 0.05 parts per million.

2. The daily output at a certain factory is $Q(L) = 900L^{\frac{1}{3}}$ units, where L denotes the size of the labor force measured in worker-hours. Currently, 1000 worker-hours of labor are used each day. Use calculus to estimate the number of additional worker-hours of labor that will be needed to increase daily output by 15 units.

   Solution. Solve for $\triangle L$ using the approximation formula

$$\triangle Q \approx Q'(L)\triangle L$$

   with $\triangle Q = 15$, $L = 1000$ and $Q'(L) = 300L^{-\frac{2}{3}}$ to get

$$15 \approx \frac{300}{\sqrt[3]{1000^2}}\triangle L$$

   or $\triangle L \approx \frac{15}{300}\sqrt[3]{1000^2} = \frac{1}{20}\sqrt[3]{10^6} = 5$ worker-hours.

3. Records indicate that x years after 1988, the average property tax on a three bedroom home in a certain community was $T(x) = 60x^{\frac{3}{2}} + 40x + 1200$ dollars. Estimate the percentage by which the property tax increased during the first half of 1992.

   Solution. First let us calculate $T'(x) = 90x^{\frac{1}{2}} + 40 = 90\sqrt{x} + 40$. Associate the number x_1 with the start of 1992 and the number x_2 with the middle of 1992. Thus $x_1 = 4$ and $x_2 = 4.5$ and we have to estimate

$$\begin{aligned}\text{Percentage change in } T &= \frac{T'(4)\triangle x}{T(4)}100 = \frac{(4.5-4)T'(4)}{T(4)}100 = \\ &= \frac{\frac{1}{2}\left(90\sqrt{4}+40\right)100}{60\cdot 4\sqrt{4}+40\cdot 4+1200} \approx 5.98 \text{ percent.}\end{aligned}$$

3. A soccer ball made of leather $\frac{1}{8}$ inch thick has an inner diameter of $8\frac{1}{2}$ inches. Estimate the volume of its leather shell.

   Solution. Think of the volume of the shell as a certain change $\triangle V$ in volume. Recall that the volume of a ball equals $V(d) = \frac{4}{3}\left(\frac{d}{2}\right)^3\pi = \frac{d^3}{6}\pi$, where d denotes the diameter of the ball. Now, use the approximation formula

$$\triangle V \approx V'(d)\triangle d$$

   with $V'(d) = \frac{1}{2}d^2\pi$, $d = 8\frac{1}{2}$ and $\triangle d = \frac{2}{8}$ to get

$$\triangle V \approx \frac{1}{2}\left(\frac{17}{2}\right)^2\pi\frac{2}{8} = \frac{289}{32}\pi \text{ cubic inches.}$$

Homework

1. At a certain factory, the daily output is $Q(K) = 600K^{\frac{1}{2}}$ units, where K denotes the capital investment measured in units of \$1000.The current capital investment is \$900000. Estimate the effect that an additional capital investment of \$800 will have on the daily output.

2. A melon in the form of a sphere has a rind $\frac{1}{5}$ inch thick and an inner diameter of 8 inches. Estimate what percentage of the total volume of the melon is rind.

3. The output at a certain factory is $Q = 600K^{\frac{1}{2}}L^{\frac{1}{3}}$ units, where K denotes the capital investment and L the size of the labor force. Estimate the percentage increase in output that will result from a 2 percent increase in the size of the labor force if capital investment is not changed

Results

1. Daily output will increase by approximately 8 units.

2. 15 percent

3. 0.67 percent

4.7 The Second Derivative

1. Find the second derivative of the function $f(x) = 5\sqrt{x} + \frac{3}{x^2} + \frac{1}{3\sqrt{x}} + \frac{1}{2}$.

   Solution. Let us find the first derivative

   $$f'(x) = \left(5\sqrt{x} + \frac{3}{x^2} + \frac{1}{3\sqrt{x}} + \frac{1}{2}\right)' = \frac{5}{2}x^{-\frac{1}{2}} - 6x^{-3} - \frac{1}{6}x^{-\frac{3}{2}}.$$

   In order to find the second derivative of f we have to differentiate the first derivative f'. Now, we get

   $$f''(x) = [f'(x)]' = \left(\frac{5}{2}x^{-\frac{1}{2}} - 6x^{-3} - \frac{1}{6}x^{-\frac{3}{2}}\right)' =$$

   $$-\frac{5}{4}x^{-\frac{3}{2}} + 18x^{-4} + \frac{1}{4}x^{-\frac{5}{2}} = -\frac{5}{4x\sqrt{x}} + \frac{18}{x^4} + \frac{1}{4x^2\sqrt{x}}.$$

2. Find the second derivative of the function $f(x) = x(2x+1)^4$.

   Solution. Since the first derivative $f'$of f equals

   $$f'(x) = \left[x(2x+1)^4\right]' = (2x+1)^4 + 8x(2x+1)^3,$$

it follows that

$$f''(x) = [f'(x)]' = \left[(2x+1)^4 + 8x(2x+1)^3\right]' =$$
$$8(2x+1)^3 + 8(2x+1)^3 + 24x(2x+1)^2 2 = 16(5x+1)(2x+1)^2.$$

3. An object moves across a straight line so that after t seconds its distance from its starting point is $D(t) = t^3 - 12t^2 + 100t + 12$ meters. Find the acceleration of the object after 3 seconds. Is the object slowing down or speeding up at this time?

   Solution. We need to find $D''(3)$. Since the the first derivative D' of D equals

   $$D'(t) = (t^3 - 12t^2 + 100t + 12)' = 3t^2 - 24t + 100$$

   it follows that

   $$D''(t) = (3t^2 - 24t + 100)' = 6t - 24.$$

   Hence $D''(3) = -6$ and we conclude that the object is slowing down.

Homework

In Problems 1 through 5 find the second derivative of the given function.

1. $f(x) = 5\sqrt[3]{x} + \dfrac{2}{x^2} + \dfrac{1}{3\sqrt{x}} - \dfrac{1}{2}$
2. $f(x) = x(3x+2)^3$
3. $f(x) = (x+1)^2 + 7$
4. $f(x) = \dfrac{x-1}{x+1}$
5. $f(x) = \dfrac{x+2}{x-2}$
6. If an object is dropped or thrown vertically, its height (in feet) after t seconds is $H(t) = -16t^2 + S_0 t + H_0$, where S_0 is the initial speed and of the object and H_0 its initial height. Derive an expression for the acceleration of the object.

Results

1. $f''(x) = -\frac{10}{9\left(\sqrt[3]{x}\right)^5} + \frac{12}{x^4} + \frac{1}{4\left(\sqrt{x}\right)^5}$.

2. $f''(x) = 324x^2 + 324x + 72.$
3. $f''(x) = 2.$
4. $f''(x) = -\frac{4}{(x+1)^3}.$
5. $f''(x) = \frac{8}{(x-2)^3}.$
6. $H''(t) = -32$

4.8 Extrema

1. Find the intervals of increase and decrease and the relative extrema of the given function.

   (a) $f(x) = x^4 + 8x^3 + 18x^2 - 8$

   (b) $f(x) = \dfrac{x^2}{x-2}$

   (c) $f(x) = \sqrt[3]{x^2}$

   Solution.

   (a) The derivative is

   $$f'(x) = 4x^3 + 24x^2 + 36x = 4x(x+3)^2$$

   which is zero when $x = 0$ and $x = -3$. Since $f(0) = -8$ and $f(-3) = 19$, the corresponding critical points are $(0, -8)$ and $(-3, 19)$. To find the intervals of increase and decrease of the function, we have to check the sign of the derivative. Look at the following table

Interval	$x < -3$	$-3 < x < 0$	$x > 0$
x	$-$	$-$	$+$
$(x+3)^2$	$+$	$+$	$+$
sign of f'	$-$	$-$	$+$

   Notice that the function f is decreasing on both sides of the critical point $(-3, 19)$,and thus $(-3, 19)$ is not a relative extremum. On the other hand, the critical point $(0, -8)$ is a relative minimum since f is increasing for the values $x > 0$.

(b) Since the denominator of $f(x)$ is zero when $x = 2$, 2 is not in the domain of f. By the quotient rule, the derivative is

$$f'(x) = \frac{2x(x-2) - x^2}{(x-2)^2} = \frac{x^2 - 4x}{(x-2)^2} = \frac{x(x-4)}{(x-2)^2}$$

which is zero when its numerator is zero, that is, when $x = 0$ and $x = 4$. Since $f(0) = 0$ and $f(4) = 8$, the corresponding critical points are $(0, 0)$ and $(4, 8)$. The derivative is also undefined at $x = 2$, but this is not a critical value since $x = 2$ is not in the domain of the function. To identify the intervals of increase and decrease, check the sign of the derivative on the intervals determined by the critical values. Since $f'(x)$ is undefined at $x = 2$, its sign for $0 < x < 2$ could be different from its sign for $2 < x < 4$, and so you will have to check these two intervals separately.

Interval	$x < 0$	$0 < x < 2$	$2 < x < 4$	$x > 4$
x	$-$	$+$	$+$	$+$
$x - 4$	$-$	$-$	$-$	$+$
$(x-2)^2$	$+$	$+$	$+$	$+$
Sign of f'	$+$	$-$	$-$	$+$

The function f increases for $x < 0$, decreases for $0 < x < 2$, decreases again for $2 < x < 4$ and increases for $x > 4$. f has a relative maximum at the critical point $(0, 0)$ and a relative minimum at the critical point $(4, 8)$.

(c) The derivative

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

is never zero, but is undefined at $x = 0$, which is in the domain of f. Thus, since $f(0) = 0$, the corresponding point $(0, 0)$ is the only critical point. To find the intervals of increase and decrease, check the sign of the derivative for $x < 0$ and $x > 0$.

Interval	$x < 0$	$x > 0$
Sign of f'	$-$	$+$

The function f decreases for $x < 0$ and increases for $x > 0$. Thus, the critical point $(0, 0)$ is a relative minimum.

2. Use the second derivative test to find the relative extrema of the function $f(x) = x + \dfrac{1}{x}$.

Solution. Since the first derivative $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$ is zero when $x = 1$ and $x = -1$, the corresponding critical points are $(1, 2)$ and $(-1, -2)$. To test these points, compute the second derivative $f''(x) = \frac{2}{x^3}$ and evaluate it at $x = 1$ and $x = -1$. Since $f''(1) = 2 > 0$ it follows that the critical point $(1, 2)$ is a relative minimum, and since $f''(-1) = -2 < 0$ it follows that the critical point $(-1, -2)$ is a relative maximum.

3. Find the absolute maximum and absolute minimum of the given function on the specified closed interval.

   (a) $f(x) = \dfrac{1}{x^2 - 9}; \quad -1 \leq x \leq 2$

   (b) $f(x) = \mid x \mid; \quad -2 \leq x \leq 3$

   Solution.

   (a) From the first derivative $f'(x) = \frac{-2x}{(x^2-9)^2}$ you see that $f'(x) = 0$ only when $x = 0$ and $f'(x)$ is undefined when $x = 3$. Since 3 is not in the domain of f, 0 is the only critical value of f. Note that 0 lies in the interval $-1 \leq x \leq 2$. Compute $f(x)$ at $x = 0$ and at endpoints $x = -1$ and $x = 2$. You get $f(0) = -\frac{1}{9}$, $f(-1) = -\frac{1}{8}$ and $f(2) = -\frac{1}{5}$. Compare these values to conclude that the absolute maximum of f on the closed interval $-1 \leq x \leq 2$ is $f(0) = -\frac{1}{9}$ and the absolute minimum is $f(2) = -\frac{1}{5}$.

   (b) Note that $f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ and $f'(x)$ is not defined when $x = 0$. Hence $(0, 0)$ is the only critical point of f. Since $f(-2) = 2$ and $f(3) = 3$, it follows that $f(0) = 0$ is the absolute minimum of f on the closed interval $-2 \leq x \leq 3$ and the absolute maximum is $f(3) = 3$.

4. A manufacturer can produce radios at a cost of \$5 apiece and estimates that if they are sold for x dollars apiece, consumers will buy $20 - x$ radios a day. At what price should the manufacturer sell the radios to maximize profit?

   Solution. Let $P(x)$ denotes the manufacturer's profit. Then $P(x) = x(20 - x) - 5(20 - x) = (x - 5)(20 - x)$ and $x \geq 5$. Since $P'(x) = 20 - x - (x - 5) = 25 - 2x$, it follows that the manufacturer should sell the radios at a price of \$12.50 apiece to maximize profit.

Homework

In Problems 1 through 4, find the intervals of increase and decrease and the relative extrema of the given function.

1. $f(x) = (x^3 - 1)^4$
2. $f(x) = 2x + \dfrac{18}{x} + 1$
3. $f(x) = 2 + \sqrt[3]{(x-1)^2}$
4. $f(x) = 1 + \sqrt[3]{x}$
5. Find constants a, b, c and d so the graph of the function $f(x) = ax^3 + bx^2 + cx + d$ has a relative maximum at $(-2, 8)$ and a relative minimum at $(1, -19)$.
6. Using the second derivative test find the relative extrema of the function $f(x) = \dfrac{2}{1+x^2}$.
7. Find the absolute maximum and absolute minimum of the function $f(x) = \dfrac{x^2+3}{x-1}$ on the closed interval $2 \leq x \leq 4$.
8. Poiseuille's law asserts that the speed of blood that is r centimeters from the central axis of an artery of radius R is $S(r) = c(R^2 - r^2)$, where c is a positive constant. Where is the speed of the blood greatest?

Results

1. f increases for $x > 1$ and decreases for $x < 1$, $(1, 0)$ is a relative minimum.
2. f increases for $x < -3$, decreases for $-3 < x < 0$, decreases for $0 < x < 3$, increases for $x > 3$. $(-3, -11)$ is a relative maximum and $(3, 13)$ relative minimum.
3. f decreases for $x < 1$ and increases for $x > 1$. $(1, 2)$ is a relative minimum.
4. f is increasing. No relative extrema.
5. $a = 2$, $b = 3$, $c = -12$, $d = -12$

6. $(0, 2)$ is a relative maximum.
7. $f(2) = 7$ is the absolute maximum, $f(3) = 6$ is the absolute minimum
8. At the central axis