ON THE EXISTENCE OF DIFFERENCE SETS IN GROUPS OF ORDER 96

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Abstract. The correspondence between a $(96, 20, 4)$ symmetric design having regular automorphism group and a difference set with the same parameters has been used to obtain difference sets in groups of order 96. Starting from eight such symmetric designs constructed by the tactical decomposition method, 55 inequivalent $(96, 20, 4)$ difference sets are distinguished. Thereby the existence of difference sets in 22 so far undecided nonabelian groups of order 96 is proved.

1. Introduction and preliminaries

A (v, k, λ) *difference set* is a k-element subset $\Delta \subseteq \Gamma$ in a group Γ of order v provided that the multiset of "differences" $xy^{-1} \mid x, y \in \Delta, x \neq y$ contains each nonidentity element of Γ exactly λ times. A difference set is called *abelian* (*cyclic, nonabelian*) if Γ has the respective property. The *development* of a difference set $\Delta \subseteq \Gamma$ is the incidence structure $dev\Delta = (\Gamma, \{\Delta g \mid g \in \Gamma\}, \in)$.

Difference sets with parameters

$$v = q^{d+1}(1 + \frac{q^{d+1} - 1}{q - 1}), \ \ k = q^{d} \frac{q^{d+1} - 1}{q - 1} \ \ and \ \lambda = q^{d} \frac{q^{d} - 1}{q - 1}, \tag{1.1}$$

where q is any prime power and d is any positive integer, form the series of so called McFarland difference sets. The first construction of the series by McFarland [7] was performed in groups of the type $E \times K$, where E denotes the elementary abelian group of order q^{d+1} and K is an arbitrary group. Later, very important generalization of this result was given by Dillon, [5]. He proved McFarland's construction to work out for any group of order v which contains an elementary abelian subgroup of order q^{d+1} in its center.

The classification of difference sets with parameters (1.1) is completed only for a few small values of d and q. Complete list of the total of 27 difference sets with parameters $(16, 6, 2)$, $d = 1$ and $q = 2$, is given in [12]. Up to equivalence there are exactly two $(45, 12, 3)$-difference sets, $d = 1$ and $q = 3$, both in abelian noncyclic group, [7]. The parameters $(64, 28, 12)$, $d = 2$ and $q = 2$, were examined in Dillon's research program [6], and by Smith and Liebler [13]. Their classification reveals that 259 out of 267 groups of order 64 admit a difference set.

Putting $d = 1$ and $q = 4$ in (1.1) we come to the parameters $(96, 20, 4)$, that is, to the groups of order 96. There are 231 such groups, seven of which are abelian. It is known, see [4], that difference set in an abelian group of order 96 exists if and only if its Sylow 2-subgroup has exponent at most 4. Consequently, abelian

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(96,20,4) difference sets exist precisely in the groups $Z_2^5 \times Z_3$, $Z_2^3 \times Z_4 \times Z_3$, and $Z_2 \times Z_4^2 \times Z_3$. (In [2], p.995, entry "No" for the group $Z_2 \times Z_4^2 \times Z_3$ is incorrect, see also [1]). Thus, our further attention turns to the existence status of difference sets in nonabelian groups of order 96. For some results on nonexistence the following theorem proves to be very useful, [5].

Theorem 1.1. *Let H be an abelian group and let G be the generalized dihedral extension of H; i.e., $G = \langle Q, H \rangle, Q^2 = 1, QhQ = h^{-1}$ for every $h \in H$. If G contains a difference set , then so does every abelian group which contains H as subgroup of index 2.*

A *symmetric block design* with parameters (v, k, λ) is a finite incidence structure $\mathcal{D} = (\mathcal{V}, \mathcal{B}, \mathcal{I})$ consisting of $|\mathcal{V}| = v$ points and $|\mathcal{B}| = v$ blocks, where each block is incident with k points and any two distinct points are incident with exactly λ common blocks. An *automorphism* of a symmetric block design $\mathcal{D}$ is a permutation on $\mathcal{V}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by $Aut\mathcal{D}$. If a subgroup $\Gamma \leq Aut\mathcal{D}$ acts regularly on $\mathcal{V}$ and $\mathcal{B}$, then $\mathcal{D}$ is called *regular* and Γ is called a *Singer group* of $\mathcal{D}$.

The close relation between difference sets and symmetric block designs goes as follows.

Theorem 1.2. *Let Γ be a finite group of order v and Δ a proper, non-empty k-element subset of Γ. Then Δ is a (v, k, λ) difference set in Γ if and only if $dev\Delta$ is a symmetric (v, k, λ) design on which Γ acts regularly.*

The details of this relation, as well as the proof of the theorem, can be found in [2], p.299.

An automorphism $\varphi \in Aut\Gamma$ is called a *multiplier* of the difference set $\Delta \subseteq \Gamma$ if $\varphi(\Delta) = g_1 \Delta g_2$ for some $g_1, g_2 \in \Gamma$. When $\varphi(\Delta) = \Delta g$ for some $g \in \Gamma$, φ is called a *right multiplier.* All multipliers of Δ form a group with the subgroup of right multipliers. Up to isomorphism the latter group is determined by the following theorem ([2], p.310).

Theorem 1.3. *Let Δ be a difference set in a group Γ and let M denote the group of all right multipliers of Δ. Then, $M \cong N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ in $Autdev\Delta$.*

Obviously, any right multiplier of Δ is an automorphism of the design $dev\Delta$.

Two difference sets Δ^1 (in Γ^1) and Δ^2 (in Γ^2) are *isomorphic* if the designs $dev\Delta^1$ and $dev\Delta^2$ are isomorphic; Δ^1 and Δ^2 are *equivalent* if there exists a group isomorphism $\varphi : \Gamma^1 \to \Gamma^2$ such that $\varphi(\Delta^1) = \Delta^2 g$ for a suitable $g \in \Gamma^2$. It is clear that equivalent difference sets Δ^1 and Δ^2 give rise to isomorphic symmetric designs $dev\Delta^1$ and $dev\Delta^2$. Also, two isomorphic difference sets having nonisomorphic groups of right multipliers are inequivalent. Nonabelian difference sets which give a symmetric design having no abelian group acting regularly on its points are called *genuinely nonabelian.*

Aiming to construct difference sets in some groups of order 96, we'll get use of the interrelation between difference sets and symmetric designs given in Theorem 1.2. In this sense we search for $(96, 20, 4)$ regular symmetric designs. For design construction we use the well known method of tactical decomposition, based on the action of an appropriate group [11]. Among admissible groups, that group is to be chosen so as to give our search sufficiently wide scope, keeping it simultaneously within our computational reach. Bearing that in mind, as well as the results on abelian (96,20,4) difference sets which suggest Z_2^4 to be "good" automorphism group for the designs leading to difference sets, we take into consideration groups of the type $Z_2^4 \cdot Z_3$ (a semidirect product of the elementary abelian group of order 16 by Z_3). We confine ourselves to their action on (96,20,4) symmetric designs in orbits of length 16 stabilized by Z_3. In this case stabilizer Z_3 has nontrivial permutation representation of degree 16 ($Z_3 \leq Z_2^4 \cdot Z_3$), which will ensure feasibility to our computation in acceptable time.

There are exactly two groups of the type $Z_2^4 \cdot Z_3$. In Section 2 we show that, under given assumptions, only one of them can act on symmetric (96,20,4) designs. From the set of all the symmetric designs obtained by the action of this group we single out those with transitive full automorphism group. There are eight such nonisomorphic designs with as many nonisomorphic full automorphism groups. In Section 3 we describe the construction of difference sets in all the subgroups of a particular full automorphism group that prove to be Singer groups of the underlying design.

The properties of the full automorphism group $AutD$ of the design D we examine by using GAP, system for computational group theory, [8]. As the number of groups we deal with is rather large, it is important to introduce systematic and clear identification marks for them. For that purpose it seems reasonable to use GAP-catalogue number (GAP-cn) appointed to each group in the GAP Library Small Groups. Its form $[m, j]$ denotes j-th group of order m in the catalogue. If m is too large and the Library doesn't contain the related group, we put $[m, -]$.

We assume that readers are familiar with the notions such as orbit matrix, indexing, etc., which are essential for the method applied and detailed in [9], for instance. Our notation is mostly in accordance with [2] and [3].

2. Construction of symmetric designs

In terms of generators and relations the semidirect products of the group Z_2^4 by Z_3 can be given as:

$$\begin{aligned} G_1 = [48, 49] = \ & \langle c, d, e, f, g \mid c^2 = d^2 = e^2 = f^2 = 1, [c, d] = [c, e] = [c, f] = 1, \\ & [d, e] = [d, f] = [e, f] = 1, g^3 = 1, c^g = c, d^g = d, e^g = ef, f^g = e\rangle, \end{aligned}$$

$$\begin{aligned} G_2 = [48, 50] = \ & \langle c, d, e, f, g \mid c^2 = d^2 = e^2 = f^2 = 1, [c, d] = [c, e] = [c, f] = 1, \\ & [d, e] = [d, f] = [e, f] = 1, g^3 = 1, c^g = cd, d^g = c, e^g = ef, f^g = e\rangle. \end{aligned}$$

We use the notation $p^q = qpq^{-1}$ for p, q arbitrary elements of a group.

We consider possible G_i-action, $i = 1, 2$, on a $(96, 20, 4)$ symmetric design in six orbits of length 16. The only orbit matrix in that case is (2.1). The entries of this matrix give a possible dispersion (regarding cardinality) of the points lying on the blocks of each block orbit into point orbits. Design construction is equivalent to indexing orbit matrix (2.1), that is determining precisely which points from every point orbit lie on a representative block of each block orbit. Preceded by making highly optimized programs, indexing is performed by using the computer. In case of the group G_1 the completion of the indexing procedure proves to be impossible, i.e. the associated designs don't exist. In what follows we give the successful construction obtained by the group G_2, pointing out what's interesting with regard to Section 3.

$$\begin{array}{cccccc|c} 16 & 16 & 16 & 16 & 16 & 16 & \\ \hline 0 & 4 & 4 & 4 & 4 & 4 & 16 \\ 4 & 0 & 4 & 4 & 4 & 4 & 16 \\ 4 & 4 & 0 & 4 & 4 & 4 & 16 \\ 4 & 4 & 4 & 0 & 4 & 4 & 16 \\ 4 & 4 & 4 & 4 & 0 & 4 & 16 \\ 4 & 4 & 4 & 4 & 4 & 0 & 16 \end{array} \tag{2.1}$$

Theorem 2.1. *There are exactly eight nonisomorphic (96,20,4) symmetric designs with* G_2 *as an automorphism group acting in six orbits of length* 16 *and with transitive full automorphism group.*

Proof. As the group acts in orbits of length 16, we need a group G_2 generators' permutation representation of degree 16 (on $\langle g\rangle$-cosets) for indexing the orbit matrix (2.1). The one used here is the following.

$$\begin{array}{l} c{=}(1\ 2)(3\ 6)(4\ 8)(5\ 7)(9\ 11)(10\ 15)(12\ 16)(13\ 14) \\ d{=}(1\ 3)(2\ 6)(4\ 14)(5\ 15)(7\ 10)(8\ 13)(9\ 16)(11\ 12) \\ e{=}(1\ 4)(2\ 8)(3\ 14)(5\ 12)(6\ 13)(7\ 16)(9\ 10)(11\ 15) \\ f{=}(1\ 5)(2\ 7)(3\ 15)(4\ 12)(6\ 10)(8\ 16)(9\ 13)(11\ 14) \\ g{=}(2\ 3\ 6)(4\ 5\ 12)(7\ 11\ 13)(8\ 15\ 9)(10\ 16\ 14) \end{array} \tag{2.2}$$

Let $1, 2, \ldots, 16$ be the points of point orbits of our design. As design representative blocks (six of them, each representing one block orbit) we take blocks stabilized by the subgroup $\langle g\rangle \leq G_2$. Therefore, these blocks are to be composed from $\langle g\rangle$-point orbits as a whole, them being 1 fixed point and 5 orbits of length three, (2.2). The related number of possibilities for a selection of 20 points is 5^5. In the procedure of indexing, on each level, every possible selection of orbit representative block is submitted to all the necessary λ-balance checkings, and this is performed by computer. The procedure finally ends up with a great number of symmetric designs constructed. For the elimination of isomorphic structures we use program Nauty [14]. It turns out that there are exactly 37 nonisomorphic symmetric designs admitting the specified action of G_2. Eight of them, denoted $D_i, i = 1, \ldots, 8$, have full automorphism group acting transitively, [14]. We give them by their six base blocks.

D_1
$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$
$1_1\ 1_8\ 1_9\ 1_{15}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 4_1\ 4_4\ 4_5\ 4_{12}\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}\ 6_1\ 6_2\ 6_3\ 6_6$
$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 4_1\ 4_2\ 4_3\ 4_6\ 5_1\ 5_4\ 5_5\ 5_{12}\ 6_1\ 6_8\ 6_9\ 6_{15}$
$1_1\ 1_7\ 1_{11}\ 1_{13}\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 3_1\ 3_8\ 3_9\ 3_{15}\ 5_1\ 5_2\ 5_3\ 5_6\ 6_1\ 6_4\ 6_5\ 6_{12}$
$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 3_1\ 3_2\ 3_3\ 3_6\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 6_1\ 6_7\ 6_{11}\ 6_{13}$
$1_1\ 1_2\ 1_3\ 1_6\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_7\ 5_{11}\ 5_{13}$
D_2
$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$
$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 3_1\ 3_2\ 3_3\ 3_6\ 4_1\ 4_4\ 4_5\ 4_{12}\ 5_1\ 5_7\ 5_{11}\ 5_{13}\ 6_1\ 6_8\ 6_9\ 6_{15}$
$1_1\ 1_8\ 1_9\ 1_{15}\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 4_1\ 4_2\ 4_3\ 4_6\ 5_1\ 5_4\ 5_5\ 5_{12}\ 6_1\ 6_7\ 6_{11}\ 6_{13}$
$1_1\ 1_7\ 1_{11}\ 1_{13}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 5_1\ 5_2\ 5_3\ 5_6\ 6_1\ 6_4\ 6_5\ 6_{12}$
$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 3_1\ 3_8\ 3_9\ 3_{15}\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 6_1\ 6_2\ 6_3\ 6_6$
$1_1\ 1_2\ 1_3\ 1_6\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}$

D_3

$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$

$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 3_1\ 3_8\ 3_9\ 3_{15}\ 4_1\ 4_4\ 4_5\ 4_{12}\ 5_1\ 5_7\ 5_{11}\ 5_{13}\ 6_1\ 6_2\ 6_3\ 6_6$

$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 5_1\ 5_2\ 5_3\ 5_6\ 6_1\ 6_7\ 6_{11}\ 6_{13}$

$1_1\ 1_7\ 1_{11}\ 1_{13}\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_2\ 3_3\ 3_6\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}\ 6_1\ 6_8\ 6_9\ 6_{15}$

$1_1\ 1_8\ 1_9\ 1_{15}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 4_1\ 4_2\ 4_3\ 4_6\ 6_1\ 6_4\ 6_5\ 6_{12}$

$1_1\ 1_2\ 1_3\ 1_6\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_4\ 5_5\ 5_{12}$

D_4

$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$

$1_1\ 1_8\ 1_9\ 1_{15}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 4_1\ 4_2\ 4_3\ 4_6\ 5_1\ 5_7\ 5_{11}\ 5_{13}\ 6_1\ 6_4\ 6_5\ 6_{12}$

$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 4_1\ 4_4\ 4_5\ 4_{12}\ 5_1\ 5_2\ 5_3\ 5_6\ 6_1\ 6_8\ 6_9\ 6_{15}$

$1_1\ 1_7\ 1_{11}\ 1_{13}\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 3_1\ 3_8\ 3_9\ 3_{15}\ 5_1\ 5_4\ 5_5\ 5_{12}\ 6_1\ 6_2\ 6_3\ 6_6$

$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 3_1\ 3_2\ 3_3\ 3_6\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 6_1\ 6_7\ 6_{11}\ 6_{13}$

$1_1\ 1_2\ 1_3\ 1_6\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}$

D_5

$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$

$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 3_1\ 3_8\ 3_9\ 3_{15}\ 4_1\ 4_2\ 4_3\ 4_6\ 5_1\ 5_7\ 5_{11}\ 5_{13}\ 6_1\ 6_4\ 6_5\ 6_{12}$

$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 5_1\ 5_2\ 5_3\ 5_6\ 6_1\ 6_7\ 6_{11}\ 6_{13}$

$1_1\ 1_7\ 1_{11}\ 1_{13}\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 3_1\ 3_2\ 3_4\ 3_6\ 5_1\ 5_4\ 5_5\ 5_{12}\ 6_1\ 6_8\ 6_9\ 6_{15}$

$1_1\ 1_8\ 1_9\ 1_{15}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 4_1\ 4_4\ 4_5\ 4_{12}\ 6_1\ 6_2\ 6_3\ 6_6$

$1_1\ 1_2\ 1_3\ 1_6\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}$

D_6

$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$

$1_1\ 1_7\ 1_{11}\ 1_{13}\ 3_1\ 3_2\ 3_3\ 3_6\ 4_1\ 4_4\ 4_5\ 4_{12}\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}\ 6_1\ 6_8\ 6_9\ 6_{15}$

$1_1\ 1_8\ 1_9\ 1_{15}\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 4_1\ 4_2\ 4_3\ 4_6\ 5_1\ 5_4\ 5_5\ 5_{12}\ 6_1\ 6_7\ 6_{11}\ 6_{13}$

$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 5_1\ 5_2\ 5_3\ 5_6\ 6_1\ 6_4\ 6_5\ 6_{12}$

$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 3_1\ 3_8\ 3_9\ 3_{15}\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 6_1\ 6_2\ 6_3\ 6_6$

$1_1\ 1_2\ 1_3\ 1_6\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_7\ 5_{11}\ 5_{13}$

D_7

$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$

$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 4_1\ 4_4\ 4_5\ 4_{12}\ 5_1\ 5_2\ 5_3\ 5_6\ 6_1\ 6_8\ 6_9\ 6_{15}$

$1_1\ 1_8\ 1_9\ 1_{15}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 5_1\ 5_4\ 5_5\ 5_{12}\ 6_1\ 6_2\ 6_3\ 6_6$

$1_1\ 1_7\ 1_{11}\ 1_{13}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 3_1\ 3_2\ 3_3\ 3_6\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}\ 6_1\ 6_4\ 6_5\ 6_{12}$

$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 3_1\ 3_8\ 3_9\ 3_{15}\ 4_1\ 4_2\ 4_3\ 4_6\ 6_1\ 6_7\ 6_{11}\ 6_{13}$

$1_1\ 1_2\ 1_3\ 1_6\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_7\ 5_{11}\ 5_{13}$

D_8

$2_1\ 2_2\ 2_3\ 2_6\ 3_1\ 3_4\ 3_5\ 3_{12}\ 4_1\ 4_7\ 4_{11}\ 4_{13}\ 5_1\ 5_8\ 5_9\ 5_{15}\ 6_1\ 6_{10}\ 6_{14}\ 6_{16}$

$1_1\ 1_2\ 1_3\ 1_6\ 3_1\ 3_8\ 3_9\ 3_{15}\ 4_1\ 4_{10}\ 4_{14}\ 4_{16}\ 5_1\ 5_7\ 5_{11}\ 5_{13}\ 6_1\ 6_4\ 6_5\ 6_{12}$

$1_1\ 1_4\ 1_5\ 1_{12}\ 2_1\ 2_8\ 2_9\ 2_{15}\ 4_1\ 4_2\ 4_3\ 4_6\ 5_1\ 5_{10}\ 5_{14}\ 5_{16}\ 6_1\ 6_7\ 6_{11}\ 6_{13}$

$1_1\ 1_7\ 1_{11}\ 1_{13}\ 2_1\ 2_{10}\ 2_{14}\ 2_{16}\ 3_1\ 3_2\ 3_3\ 3_6\ 5_1\ 5_4\ 5_5\ 5_{12}\ 6_1\ 6_8\ 6_9\ 6_{15}$

$1_1\ 1_8\ 1_9\ 1_{15}\ 2_1\ 2_7\ 2_{11}\ 2_{13}\ 3_1\ 3_{10}\ 3_{14}\ 3_{16}\ 4_1\ 4_4\ 4_5\ 4_{12}\ 6_1\ 6_2\ 6_3\ 6_6$

$1_1\ 1_{10}\ 1_{14}\ 1_{16}\ 2_1\ 2_4\ 2_5\ 2_{12}\ 3_1\ 3_7\ 3_{11}\ 3_{13}\ 4_1\ 4_8\ 4_9\ 4_{15}\ 5_1\ 5_2\ 5_3\ 5_6$

The points of the design are denoted by $I_1, I_2, \ldots, I_{16}$, $I = 1, 2, \ldots, 6$ as accustomed. The subgroup $\langle c, d, e, f \rangle \leq G_2$ generates all the blocks of the designs. ¥

Remark 2.1. *A computer program by V. Tonchev* [15] *computes the orders as well as the generators of the full automorphism groups* $AutD_i, i = 1, \ldots, 8$. *The generators are obtained in the form of permutation representation of degree 96, which we will need in Section 3. The essentials on these groups are given in Tables*

1 through 8. Their orders and GAP-cns of $AutD_2$, $AutD_5$, and $AutD_7$ show that $AutD_i, i = 1, \ldots, 8$, are mutually nonisomorphic. Thus, the designs $D_i, i = 1, \ldots, 8$, are selfdual.

TABLE 1. *Generators of $AutD_1$,GAP-cn :[552960,–]*

r1=(7,8,29)(9,10,42)(11,17,58)(12,91,18)(13,19,72)(14,86,20)(15,21,89)(16,90,22)(23,30,68)(24,84,31)
(25,32,81)(26,82,33)(27,34,83)(28,39,35)(36,46,52)(37,48,80)(38,74,53)(40,59,88)(41,67,63)(43,93,49)
(44,50,94)(45,87,51)(47,73,78)(54,85,60)(55,61,64)(56,57,62)(65,75,92)(66,69,79)(70,95,76)(71,77,96)
r2=(5,6)(7,9)(8,42)(10,29)(11,67)(12,84)(13,19)(14,27)(15,64)(16,90)(17,41)(18,31)(20,34)(21,61)
(23,68)(24,91)(25,43)(26,87)(28,39)(32,49)(33,51)(36,52)(38,74)(44,77)(45,82)(47,73)(48,80)(50,71)
(54,85)(55,89)(56,95)(57,70)(58,63)(59,88)(62,76)(65,92)(66,69)(81,93)(83,86)(94,96)
r3=(4,5)(8,29)(9,22)(10,90)(11,30)(12,91)(13,33)(14,54)(15,21)(16,42)(17,23)(19,82)(20,85)(24,80)
(25,50)(26,72)(27,34)(31,37)(32,44)(35,39)(38,53)(40,59)(41,63)(43,95)(45,51)(46,52)(48,84)(49,76)
(55,75)(56,71)(57,96)(58,68)(60,86)(61,65)(62,77)(64,92)(69,79)(70,93)(73,78)(81,94)
r4=(3,4)(8,29)(9,10)(12,81)(13,59)(15,56)(16,28)(17,58)(18,25)(19,40)(20,86)(21,62)(22,39)(23,53)
(24,43)(27,34)(30,74)(31,93)(32,91)(33,82)(35,90)(36,46)(37,65)(38,68)(44,94)(48,92)(49,84)(51,87)
(54,66)(55,95)(57,89)(60,69)(61,70)(63,67)(64,76)(71,77)(72,88)(73,78)(75,80)(79,85)
r5=(2,3)(8,29)(10,42)(11,31)(12,41)(13,19)(14,61)(15,34)(16,90)(17,84)(18,67)(20,64)(21,27)(23,48)
(24,58)(25,32)(28,36)(30,37)(33,82)(35,46)(38,69)(39,52)(40,78)(43,49)(44,50)(45,51)(47,88)(53,79)
(54,65)(55,86)(56,62)(59,73)(60,75)(63,91)(66,74)(68,80)(71,77)(76,95)(83,89)(85,92)
r6=(2,11,67,47,41,74)(3,26,81,66,50,65)(4,31,94,54,91,61)(5,93,18,14,32,88)(6,96,87,83,45,89)
(9,90,39)(10,22,35)(12,70,25,37,44,72)(13,80,86,92,24,20)(15,77)(16,28,42)(17,78,68,23,73,38)
(19,69,55,49,79,95)(21,71,34,62,56,27)(30,63)(33,75,84,64,43,40)(36,52,46)(48,76,85,59,82,60)(51,57)
(53,58)
r7=(1,2)(5,6)(7,11)(8,17)(9,67)(10,63)(12,24)(13,54)(14,45)(15,61)(16,68)(18,31)(19,85)(20,51)
(21,64)(22,30)(23,90)(25,93)(26,83)(27,82)(28,38)(29,58)(32,49)(33,34)(35,53)(36,73)(39,74)(40,79)
(41,42)(43,81)(44,77)(46,78)(47,52)(50,96)(55,89)(56,70)(57,95)(59,66)(60,72)(62,76)(69,88)(71,94)
(84,91)(86,87)

TABLE 2. *Generators of $AutD_2$,GAP-cn :[576,5603]*

r1=(2,3,6)(4,5,12)(7,11,13)(8,15,9)(10,16,14)(18,19,22)(20,21,28)(23,27,29)(24,31,25)(26,32,30)
(34,35,38)(36,37,44)(39,43,45)(40,47,41)(42,48,46)(50,51,54)(52,53,60)(55,59,61)(56,63,57)(58,64,62)
(66,67,70)(68,69,76)(71,75,77)(72,79,73)(74,80,78)(82,83,86)(84,85,92)(87,91,93)(88,95,89)(90,96,94)
r2=(2,10,6,14,3,16)(4,9,12,15,5,8)(7,11,13)(17,81)(18,90,22,94,19,96)(20,89,28,95,21,88)
(23,91,29,87,27,93)(24,84,25,92,31,85)(26,86,30,83,32,82)(33,65)(34,74,38,78,35,80)
(36,73,44,79,37,72)(39,75,45,71,43,77)(40,68,41,76,47,69)(42,70,46,67,48,66)(50,58,54,62,51,64)
(52,57,60,63,53,56)(55,59,61)
r3=(1,2)(3,6)(4,8)(5,7)(9,11)(10,15)(12,16)(13,14)(17,18)(19,22)(20,24)(21,23)(25,27)(26,31)(28,32)
(29,30)(33,34)(35,38)(36,40)(37,39)(41,43)(42,47)(44,48)(45,46)(49,50)(51,54)(52,56)(53,55)(57,59)
(58,63)(60,64)(61,62)(65,66)(67,70)(68,72)(69,71)(73,75)(74,79)(76,80)(77,78)(81,82)(83,86)(84,88)
(85,87)(89,91)(90,95)(92,96)(93,94)
r4=(1,17,33,49,65,81)(2,18,34,50,66,82)(3,19,35,51,67,83)(4,20,36,52,68,84)(5,21,37,53,69,85)
(6,22,38,54,70,86)(7,23,39,55,71,87)(8,24,40,56,72,88)(9,25,41,57,73,89)(10,26,42,58,74,90)
(11,27,43,59,75,91)(12,28,44,60,76,92)(13,29,45,61,77,93)(14,30,46,62,78,94)(15,31,47,63,79,95)
(16,32,48,64,80,96)

TABLE 3. *Generators of* $AutD_3$,*GAP-cn :[1728,47858]*

r1=(5,6,22)(7,8,27)(10,11,37)(12,13,42)(14,16,49)(15,17,47)(18,20,60)(19,21,58)(23,25,84)(24,50,65)
(26,63,66)(29,31,46)(30,32,71)(33,35,70)(34,36,81)(38,40,95)(39,74,54)(41,64,67)(43,86,45)(44,85,82)
(48,87,72)(51,89,75)(52,53,76)(55,90,73)(56,57,96)(59,68,79)(61,78,93)(62,94,91)(69,80,83)(77,92,88)
r2=(3,4)(5,6,22)(7,24,27,65,8,50)(9,28)(10,11,37)(12,39,42,54,13,74)(14,17,49,15,16,47)
(18,48,60,72,20,87)(19,51,58,75,21,89)(23,63,84,26,25,66)(29,32,46,30,31,71)(33,62,70,91,35,94)
(34,44,81,82,36,85)(38,64,95,41,40,67)(43,88,45,92,86,77)(52,68,76,59,53,79)(55,57,73,56,90,96)
(61,80,93,69,78,83)
r3=(2,9,28)(5,84,26)(6,23,63)(10,86,77)(11,45,92)(12,33,34)(13,35,36)(14,96,59)(15,76,55)(16,56,68)
(17,52,90)(22,25,66)(24,65,50)(29,93,41)(30,95,69)(31,61,64)(32,38,80)(37,43,88)(39,82,94)(40,83,71)
(42,70,81)(44,91,74)(46,78,67)(47,53,73)(48,72,87)(49,57,79)(51,75,89)(54,85,62)
r4=(1,2)(3,9)(4,28)(5,10)(6,11)(7,54)(8,39)(12,65)(13,24)(14,29)(15,30)(16,31)(17,32)(18,82)(19,91)
(20,44)(21,62)(22,37)(23,41)(25,64)(26,38)(27,74)(33,75)(34,72)(35,51)(36,48)(40,63)(42,50)(43,68)
(45,59)(46,49)(47,71)(52,77)(53,92)(55,61)(56,69)(57,80)(58,94)(60,85)(66,95)(67,84)(70,89)(73,93)
(76,88)(78,90)(79,86)(81,87)(83,96)
r5=(1,5)(2,10)(3,14)(4,15)(6,22)(7,23)(8,24)(9,29)(11,37)(12,38)(13,39)(16,49)(17,47)(18,56)(19,52)
(20,51)(21,48)(25,63)(26,65)(27,66)(28,30)(31,46)(32,71)(33,45)(34,61)(35,44)(36,62)(40,64)(41,54)
(42,67)(43,80)(50,84)(53,90)(55,72)(57,68)(58,73)(59,75)(60,79)(69,82)(70,83)(74,95)(76,87)(77,91)
(78,92)(81,88)(85,86)(89,96)(93,94)

TABLE 4. *Generators of* $AutD_4$,*GAP-cn :[3456,−]*

r1=(17,24)(18,20)(19,29)(21,32)(22,30)(23,28)(25,31)(26,27)(33,34)(35,38)(36,40)(37,39)(41,43)
(42,47)(44,48)(45,46)(49,58)(50,63)(51,55)(52,57)(53,54)(56,59)(60,61)(62,64)(65,79)(66,74)(67,69)
(68,75)(70,71)(72,73)(76,78)(77,80)(81,91)(82,89)(83,92)(84,95)(85,94)(86,96)(87,93)(88,90)
r2=(17,25)(18,27)(19,32)(20,26)(21,29)(22,28)(23,30)(24,31)(33,38)(34,35)(36,45)(37,42)(39,47)
(40,46)(41,44)(43,48)(49,62)(50,61)(51,52)(53,59)(54,56)(55,57)(58,64)(60,63)(65,72)(66,68)(67,77)
(69,80)(70,78)(71,76)(73,79)(74,75)(81,87)(82,85)(83,90)(84,96)(86,95)(88,92)(89,94)(91,93)
r3=(2,3,6)(4,5,12)(7,11,13)(8,15, 9)(10,16,14)(18,19,22)(20,21,28)(23,27,29)(24,31,25)(26,32,30)
(34,35,38)(36,37,44)(39,43,45)(40,47,41)(42,48,46)(50,51,54)(52,53,60)(55,59,61)(56,63,57)
(58,64,62)(66,67,70)(68,69,76)(71,75,77)(72,79,73)(74,80,78)(82,83,86)(84,85,92)(87,91,93)
(88,95,89)(90,96,94)
r4=(2,7,16)(3,11,14)(4,12,5)(6,13,10)(18,22,19)(20,30,29)(21,26,23)(27,28,32)(33,81,49)(34,91,58)
(35,93,64)(36,90,59)(37,96,61)(38,87,62)(39,86,60)(40,88,56)(41,89,57)(42,84,50)(43,82,52)
(44,94,55)(45,83,53)(46,92,54)(47,95,63)(48,85,51)(66,78,69)(67,74,76)(68,70,80)(71,77,75)
r5=(1,2)(3,6)(4,8)(5,7)(9,11)(10,15)(12,16)(13,14)(17,18)(19,22)(20,24)(21,23)(25,27)(26,31)(28,32)
(29,30)(33,34)(35,38)(36,40)(37,39)(41,43)(42,47)(44,48)(45,46)(49,50)(51,54)(52,56)(53,55)(57,59)
(58,63)(60,64)(61,62)(65,66)(67,70)(68,72)(69,71)(73,75)(74,79)(76,80)(77,78)(81,82)(83,86)(84,88)
(85,87)(89,91)(90,95)(92,96)(93,94)
r6=(1,17,65)(2,20,74)(3,21,80)(4,18,75)(5,19,77)(6,28,78)(7,30,76)(8,24,72)(9,25,73)(10,27,66)
(11,26,68)(12,22,71)(13,32,69)(14,23,70)(15,31,79)(16,29,67)(34,37,46)(35,44,42)(36,48,38)(39,43,45)
(50,64,55)(51,62,59)(52,53,60)(54,58,61)(82,83,86)(84,93,94)(85,87,90)(91,96,92)
r7=(1,33)(2,34,3,35,6,38)(4,43,5,45,12,39)(7,46,11,42,13,48)(8,41,15,40,9,47)(10,36,16,37,14,44)(17,49)
(18,52,19,53,22,60)(20,58,21,64,28,62)(23,55,27,59,29,61)(24,57,31,56,25,63)(26,50,32,51,30,54)(65,81)
(66,90,67,96,70,94)(68,84,69,85,76,92)(71,86,75,82,77,83)(72,89,79,88,73,95)(74,91,80,93,78,87)

TABLE 5. *Generators of* $AutD_5$*,GAP-cn :[576,8654]*

r1=(2,3,6)(4,5,12)(7,11,13)(8,15,9)(10,16,14)(18,19,22)(20,21,28)(23,27,29)(24,31,25)(26,32,30)
(34,35,38)(36,37,44)(39,43,45)(40,47,41)(42,48,46)(50,51,54)(52,53,60)(55,59,61)(56,63,57)(58,64,62)
(66,67,70)(68,69,76)(71,75,77)(72,79,73)(74,80,78)(82,83,86)(84,85,92)(87,91,93)(88,95,89)(90,96,94)
r2=(2,10,3,16,6,14)(4,12,5)(7,9,11,8,13,15)(17,81)(18,90,19,96,22,94)(20,92,21,84,28,85)
(23,89,27,88,29,95)(24,93,31,87,25,91)(26,83,32,86,30,82)(34,42,35,48,38,46)(36,44,37)
(39,41,43,40,45,47)(49,65)(50,74,51,80,54,78)(52,76,53,68,60,69)(55,73,59,72,61,79)
(56,77,63,71,57,75)(58,67,64,70,62,66)
r3=(1,2)(3,6)(4,8)(5,7)(9,11)(10,15)(12,16)(13,14)(17,18)(19,22)(20,24)(21,23)(25,27)(26,31)(28,32)
(29,30)(33,34)(35,38)(36,40)(37,39)(41,43)(42,47)(44,48)(45,46)(49,50)(51,54)(52,56)(53,55)(57,59)
(58,63)(60,64)(61,62)(65,66)(67,70)(68,72)(69,71)(73,75)(74,79)(76,80)(77,78)(81,82)(83,86)(84,88)
(85,87)(89,91)(90,95)(92,96)(93,94)
r4=(1,17,49,33,65,81)(2,18,50,34,66,82)(3,19,51,35,67,83)(4,23,57,36,71,89)(5,27,56,37,75,88)
(6,22,54,38,70,86)(7,25,52,39,73,84)(8,21,59,40,69,91)(9,20,55,41,68,87)(10,32,62,42,80,94)
(11,24,53,43,72,85)(12,29,63,44,77,95)(13,31,60,45,79,92)(14,26,64,46,74,96)(15,28,61,47,76,93)
(16,30,58,48,78,90)

TABLE 6. *Generators of* $AutD_6$*,GAP-cn :[864,4678]*

r1=(2,3,6)(4,5,12)(7,11,13)(8,15,9)(10,16,14)(18,19,22)(20,21,28)(23,27,29)(24,31,25)(26,32,30)
(34,35,38)(36,37,44)(39,43,45)(40,47,41)(42,48,46)(50,51,54)(52,53,60)(55,59,61)(56,63,57)(58,64,62)
(66,67,70)(68,69,76)(71,75,77)(72,79,73)(74,80,78)(82,83,86)(84,85,92)(87,91,93)(88,95,89)(90,96,94)
r2=(2,7,16)(3,11,14)(4,12,5)(6,13,10)(17,49,81)(18,55,96)(19,59,94)(20,60,85)(21,52,92)(22,61,90)
(23,64,82)(24,56,88)(25,57,89)(26,54,93)(27,62,83)(28,53,84)(29,58,86)(30,51,91)(31,63,95)(32,50,87)
(34,39,48)(35,43,46)(36,44,37)(38,45,42)(66,71,80)(67,75,78)(68,76,69)(70,77,74)
r3=(1,2)(3,6)(4,8)(5,7)(9,11)(10,15)(12,16)(13,14)(17,18)(19,22)(20,24)(21,23)(25,27)(26,31)(28,32)
(29,30)(33,34)(35,38)(36,40)(37,39)(41,43)(42,47)(44,48)(45,46)(49,50)(51,54)(52,56)(53,55)(57,59)
(58,63)(60,64)(61,62)(65,66)(67,70)(68,72)(69,71)(73,75)(74,79)(76,80)(77,78)(81,82)(83,86)(84,88)
(85,87)(89,91)(90,95)(92,96)(93,94)
r4=(1,17,33,49,65,81)(2,18,34,50,66,82)(3,22,35,54,67,86)(4,20,36,52,68,84)(5,28,37,60,69,92)
(6,19,38,51,70,83)(7,32,39,64,71,96)(8,24,40,56,72,88)(9,31,41,63,73,95)(10,27,42,59,74,91)
(11,26,43,58,75,90)(12,21,44,53,76,85)(13,30,45,62,77,94)(14,29,46,61,78,93)(15,25,47,57,79,89)
(16,23,48,55,80,87)

TABLE 7. *Generators of* $AutD_7$*,GAP-cn :[576,5126]*

r1=(2,3,6)(4,5,12)(7,11,13)(8,15, 9)(10,16,14)(18,19,22)(20,21,28)(23,27,29)(24,31,25)(26,32,30)
(34,35,38)(36,37,44)(39,43,45)(40,47,41)(42,48,46)(50,51,54)(52,53,60)(55,59,61)(56,63,57)(58,64,62)
(66,67,70)(68,69,76)(71,75,77)(72,79,73)(74,80,78)(82,83,86)(84,85,92)(87,91,93)(88,95,89)(90,96,94)
r2=(2,10,6,14,3,16)(4,9,12,15,5,8)(7,11,13)(17,81)(18,90,22,94,19,96)(20,89,28,95,21,88)
(23,91,29,87,27,93)(24,84,25,92,31,85)(26,86,30,83,32,82)(33,65)(34,74,38,78,35,80)
(36,73,44,79,37,72)(39,75,45,71,43,77)(40,68,41,76,47,69)(42,70,46,67,48,66)(50,58,54,62,51,64)
(52,57,60,63,53,56)(55,59,61)
r3=(1,2)(3,6)(4,8)(5,7)(9,11)(10,15)(12,16)(13,14)(17,18)(19,22)(20,24)(21,23)(25,27)(26,31)(28,32)
(29,30)(33,34)(35,38)(36,40)(37,39)(41,43)(42,47)(44,48)(45,46)(49,50)(51,54)(52,56)(53,55)(57,59)
(58,63)(60,64)(61,62)(65,66)(67,70)(68,72)(69,71)(73,75)(74,79)(76,80)(77,78)(81,82)(83,86)(84,88)

(85,87)(89,91)(90,95)(92,96)(93,94)
r4=(1,17,65,49,33,81)(2,18,66,50,34,82)(3,22,67,54,35,86)(4,24,68,56,36,88)(5,25,69,57,37,89)
(6,19,70,51,38,83)(7,27,71,59,39,91)(8,20,72,52,40,84)(9,21,73,53,41,85)(10,32,74,64,42,96)
(11,23,75,55,43,87)(12,31,76,63,44,95)(13,29,77,61,45,93)(14,30,78,62,46,94)(15,28,79,60,47,92)
(16,26,80,58,48,90)

TABLE 8. *Generators of* $AutD_8$*,GAP-cn :[138240,−]*

r1=(4,12,5)(7,8,16)(9,10,13)(11,15,14)(18,19,22)(20,32,23)(21,30,27)(24,25,31)(26,29,28)(34,48,39)
(35,46,43)(36,37,44)(38,42,45)(49,65,81)(50,74,93)(51,80,87)(52,70,88)(53,66,95)(54,78,91)(55,79,96)
(56,69,83)(57,68,82)(58,77,84)(59,73,94)(60,67,89)(61,72,90)(62,75,92)(63,76,86)(64,71,85)
r2=(17,18)(19,22)(20,24)(21,23)(25,27)(26,31)(28,32)(29,30)(33,36)(34,40)(35,46)(37,44)(38,45)
(39,48)(41,42)(43,47)(49,61)(50,62)(51,56)(52,54)(53,57)(55,59)(58,60)(63,64)(65,73)(66,75)(67,80)
(68,74)(69,77)(70,76)(71,78)(72,79)(81,96)(82,92)(83,89)(84,87)(85,88)(86,91)(90,94)(93,95)
r3=(4,13)(5,7)(8,14)(9,16)(10,15)(11,12)(20,29)(21,23)(24,30)(25,32)(26,31)(27,28)(33,49)(34,50)
(35,51)(36,61)(37,55)(38,54)(39,53)(40,62)(41,64)(42,63)(43,60)(44,59)(45,52)(46,56)(47,58)(48,57)
(65,81)(66,82)(67,83)(68,93)(69,87)(70,86)(71,85)(72,94)(73,96)(74,95)(75,92)(76,91)(77,84)(78,88)
(79,90)(80,89)
r4=(3,6)(4,12)(8,16)(9,14)(10,15)(11,13)(19,22)(20,27)(21,23)(24,25)(28,29)(30,32)(35,38)(37,44)
(39,48)(41,47)(42,43)(45,46)(51,54)(52,56)(53,57)(55,59)(58,64)(60,63)(65,81)(66,82)(67,86)(68,95)
(69,88)(70,83)(71,84)(72,90)(73,96)(74,93)(75,92)(76,89)(77,85)(78,87)(79,94)(80,91)
r5=(2,3)(5,12)(7,11)(8,14)(9,10)(15,16)(18,19)(20,29)(21,27)(23,28)(25,31)(26,32)(34,35)(36,37)
(39,46)(40,47)(42,45)(43,48)(50,51)(52,63)(53,56)(55,61)(57,60)(58,62)(65,81)(66,83)(67,82)(68,89)
(69,95)(70,86)(71,85)(72,96)(73,94)(74,87)(75,84)(76,88)(77,92)(78,91)(79,90)(80,93)
r6=(2,4,3,5,6,12)(7,13,11)(8,14,15,10, 9,16)(17,33)(18,44,19,36,22,37)(20,46,21,42,28,48)
(23,45,27,39,29,43)(24,47,31,41,25,40)(26,38,32,34,30,35)(50,64,51,62,54,58)(52,56,53,63,60,57)
(55,61,59)(65,81)(66,83,67,86,70,82)(68,89,69,88,76,95)(71,93,75,87,77,91)(72,96,79,94,73,90)
(74,84,80,85,78,92)
r7=(1,2)(4,16)(5,7)(8,12)(9,13)(11,14)(17,18)(20,25)(24,27)(26,31)(28,30)(29,32)(33,34)(36,40)
(37,48)(39,44)(41,42)(43,47)(49,50)(53,59)(55,57)(58,60)(61,62)(63,64)(65,82)(66,81)(67,83)(68,90)
(69,84)(70,86)(71,88)(72,95)(73,92)(74,94)(75,96)(76,91)(77,87)(78,85)(79,93)(80,89)
r8=(1,17)(2,18)(3,19)(4,24)(5,31)(6,22)(7,26)(8,20)(9,28)(10,23)(11,32)(12,25)(13,30)(14,29)(15,21)
(16,27)(36,40)(37,47)(39,42)(41,44)(43,48)(45,46)(52,56)(53,63)(55,58)(57,60)(59,64)(61,62)(65,81)
(66,82)(67,83)(68,88)(69,95)(70,86)(71,90)(72,84)(73,92)(74,87)(75,96)(76,89)(77,94)(78,93)(79,85)
(80,91)

3. Difference sets

Given the permutation representation of degree 96 of the group $F_i = AutD_i, i = 1, \ldots, 8$, and knowing that F_i acts transitively on symmetric design D_i, the set $\{1, 2, \ldots, 96\}$ can be taken as the set of points of D_i and, up to isomorphism, D_i can be presented by one base block $b_i, i = 1, \ldots, 8$. One choice of the design representative $b_i, i = 1, \ldots, 8$, we give in (3.1). All the blocks of the design D_i can be obtained by the action of the group F_i (or any of its transitive subgroups) on $b_i, i = 1, \ldots, 8$. Starting from b_i, it is possible to construct difference set in every subgroup $H \leq F_i$ which is a Singer group of D_i (Theorem 1.2). Hence, the first task regarding difference sets construction is to determine the Singer subgroups of F_i. To this end, we use the permutation representations given in Tables 1 through

8 and GAP ([8]). Provided the applied algorithms are optimized with regard to the structure of the group F_{i}, GAP proves to be very powerful tool.

$$
\begin{aligned}
b_1 &=\{1,2,4,5,6,15,21,28,35,38,39,53,55,61,64,65,74,75,89,92\}\\
b_2 &=\{1,10,14,16,33,34,35,38,49,52,53,60,65,71,75,77,81,88,89,95\}\\
b_3 &=\{1,2,3,4,9,15,17,26,33,35,39,47,54,56,57,63,66,70,74,96\}\\
b_4 &=\{1,8,9,15,33,42,46,48,49,50,51,54,65,71,75,77,81,84,85,92\}\\
b_5 &=\{1,10,14,16,33,40,41,47,49,50,51,54,65,71,75,77,81,84,85,92\}\\
b_6 &=\{1,7,11,13,33,34,35,38,49,52,53,60,65,74,78,80,81,88,89,95\}\\
b_7 &=\{1,10,14,16,33,39,43,45,49,52,53,60,65,66,67,70,81,88,89,95\}\\
b_8 &=\{1,2,3,6,33,40,41,47,49,58,62,64,65,71,75,77,81,84,85,92\}
\end{aligned}
\tag{3.1}
$$

In this manner we find that the group $F_1 = AutD_1$ of order 552960 has 18 nonisomorphic subgroups that act regularly on D_1. These subgroups we primarily get in the form of permutation representation of degree 96 as well. In terms of generators and relations, up to isomorphism, they can be given as:

$$
\begin{aligned}
H_{[96,13]} = {} & \langle x,y,z,t,a \mid x^4 = y^2 = z^2 = [x,y] = [y,z] = 1,\ x^{\mathsf{z}} = xy, t^2 = 1,\\
& x^{\mathsf{t}} = xz, [y,t] = [z,t] = 1, a^3 = [a,x] = [a,y] = [a,z] = 1, a^{\mathsf{t}} = a^{-1} \rangle,\\
H_{[96,41]} = {} & \langle x,y,z,t,a \mid x^4 = y^2 = z^2 = [x,y] = [y,z] = 1,\ x^{\mathsf{z}} = xy, t^2 = 1,\\
& x^{\mathsf{t}} = xz, [y,t] = [z,t] = 1, a^3, a^{\mathsf{x}} = a^{-1}, [a,y] = [a,z] = [a,t] = 1 \rangle,\\
H_{[96,64]} = {} & \langle x,y,z,a \mid x^4 = y^4 = z^2 = [x,y] = 1,\ x^{\mathsf{z}} = y, y^{\mathsf{z}} = x, a^3 = 1,\\
& x^{\mathsf{a}} = x^{-1}y^{-1}, y^{\mathsf{a}} = x, a^{\mathsf{z}} = a^{-1} \rangle,\\
H_{[96,70]} = {} & \langle x,y,z,t,a \mid x^2 = y^2 = z^2 = t^2 = [x,y] = [x,z] = [x,t] = [y,z] = 1,\\
& [y,t] = [z,t] = a^6 = 1, x^{\mathsf{a}} = yt, y^{\mathsf{a}} = xz, z^{\mathsf{a}} = xyz, t^{\mathsf{a}} = yzt \rangle,\\
H_{[96,71]} = {} & \langle x,y,z,a \mid x^4 = y^4 = z^2 = [x,y] = 1,\ x^{\mathsf{z}} = xy^2, y^{\mathsf{z}} = x^2y^{-1},\\
& a^3 = 1,\ x^{\mathsf{a}} = x^{-1}y^{-1},\ y^{\mathsf{a}} = x, [a,z] = 1 \rangle,\\
H_{[96,160]} = {} & \langle x,y,z,t,v,a \mid x^2 = y^2 = z^2 = t^2 = v^2 = [x,y] = [x,z] = [x,t] = 1,\\
& [y,z] = [y,t] = [z,t] = 1, a^3 = 1, (av)^2, x^{\mathsf{v}} = yt, y^{\mathsf{v}} = xz,\\
& z^{\mathsf{v}} = t, t^{\mathsf{v}} = z, [a,x] = [a,y] = [a,z] = [a,t] = 1 \rangle,\\
H_{[96,167]} = {} & \langle x,y,z,t,a \mid x^2 = y^2 = z^2 = t^2 = [x,y] = [x,z] = [x,t] = [y,z] = 1,\\
& [y,t] = [z,t] = 1, a^6 = 1, x^{\mathsf{a}} = y, y^{\mathsf{a}} = x, z^{\mathsf{a}} = xyz, t^{\mathsf{a}} = yzt \rangle,\\
H_{[96,185]} = {} & \langle x,y,z,a \mid x^4 = y^4 = 1, x^{\mathsf{y}} = x^{-1}, z^2 = [z,x] = 1, y^{\mathsf{z}} = x^2y^{-1},\\
& a^3 = 1, [a,x] = 1, ya = a^{-1}x^2zy^{-1}, z^{\mathsf{a}} = x^2y^2 \rangle,\\
H_{[96,186]} = {} & S_4 \times C_4 = \langle x,y,z,a \mid x^4 = y^2 = z^2 = [x,y] = [x,z] = (zy)^3 = 1,\\
& a^3 = 1, [a,x] = 1, (za)^2 = (zya)^2 = 1 \rangle,\\
H_{[96,188]} = {} & \langle x,y,a \mid x^8 = 1, y^2 = x^4, x^{\mathsf{y}} = x^{-1}, a^3 = 1, a^{\mathsf{y}} = a^{-1}, xa = a^{-1}x^3 \rangle,\\
H_{[96,190]} = {} & \langle x,y,a \mid x^8 = y^2 = 1, x^{\mathsf{y}} = x^5, a^3 = 1, (xa)^2 = 1, [y,a] = 1 \rangle,\\
H_{[96,194]} = {} & \langle x,y,z,a \mid x^4 = y^2 = z^2 = [x,y] = [y,z] = 1, x^{\mathsf{z}} = xy, a^3 = 1,\\
& a^{\mathsf{x}} = a^{-1}, z^{\mathsf{a}} = y \rangle \times \langle t \mid t^2 = 1 \rangle,\\
H_{[96,195]} = {} & \langle x,y,z,t,v,a \mid x^2 = y^2 = z^2 = t^2 = [x,y] = [x,z] = [x,t] = 1,\\
& [y,z] = [y,t] = [z,t] = 1, v^2 = a^3 = (va)^2 = 1,\ x^{\mathsf{v}} = yt,\\
& y^{\mathsf{v}} = xz, z^{\mathsf{v}} = t, t^{\mathsf{v}} = z, x^{\mathsf{a}} = xz, y^{\mathsf{a}} = yt, z^{\mathsf{a}} = t, t^{\mathsf{a}} = zt \rangle,\\
H_{[96,196]} = {} & \langle x,y,z,t,a \mid x^2 = y^2 = z^2 = t^2 = [x,y] = [x,z] = [x,t] = [y,z] = 1,\\
& [y,t] = [z,t] = 1, a^6 = xt,\ x^{\mathsf{a}} = xz,\ y^{\mathsf{a}} = yt, z^{\mathsf{a}} = t, t^{\mathsf{a}} = zt \rangle,
\end{aligned}
$$

$$H_{[96,197]} = A_4 \times D_8 = \langle x,y,a \mid x^2=y^2=a^3=1, x^{\mathsf{a}}=y, y^{\mathsf{a}}=xy\rangle \times \times\langle z,t \mid z^4=t^2=(tz)^2=1\rangle,$$

$$H_{[96,226]} = S_4 \times Z_2^2 = \langle x,y,a \mid x^2=y^2=[x,y]=a^3=(xy)^3=(ya)^2=1, (xya)^2=1\rangle \times \langle z,t \mid z^2=t^2=[z,t]=1\rangle,$$

$$H_{[96,227]} = \langle x,y,z,t,a \mid x^2=y^2=z^2=t^2=[x,y]=[x,z]=[x,t]=[y,z]=1, [y,t]=[z,t]=1, v^2=a^3=(va)^2=1,\ x^{\mathsf{v}}=y,\ y^{\mathsf{v}}=x, z^{\mathsf{v}}=t, t^{\mathsf{v}}=z, x^{\mathsf{a}}=xy, y^{\mathsf{a}}=x, z^{\mathsf{a}}=t, t^{\mathsf{a}}=zt\rangle,$$

$$H_{[96,228]} = \langle x,y,z,t,a \mid x^2=y^2=z^2=t^2=[x,y]=[x,z]=[x,t]=[y,z]=1, [y,t]=[z,t]=1, a^6=1,\ x^{\mathsf{a}}=xz,\ y^{\mathsf{a}}=yt, z^{\mathsf{a}}=t, t^{\mathsf{a}}=zt\rangle.$$

As an index label for these Singer groups we use the appropriate GAP-cn.

Identifying the points of the design D_{i} with the elements of its Singer group, yielding a difference set corresponding to b_{i}, can be performed as described in [10] and [16]. That procedure can be done by GAP as well. For instance, the following sequence of GAP-commands reveals the difference set in the group $H_{[96,13]} \le F_1$.

```
A:=Group(permutation representation of generators, Table 9 );
ds:=[points of base block b1];
F:=FreeGroup(5);a:=F.1;x:=F.2;y:=F.3;z:=F.4;t:=F.5;
H:=F/[x^4,y^2,Comm(x,y),z^2,z*x*z*y*x^-1,Comm(z,y),t^2,
t*x*t*z*x^-1, Comm(t,y), Comm(t,z), a^3,Comm(a,x),Comm(a,y),
Comm(a,z), t*a*t*a];(quotient algorithm, H ≅ H[96,13])
iso:=IsomorphismGroups(H,A); delta:=[];for d in ds do for g in H
do if 1^Image(iso,g)=d then AddSet(delta,g);fi;od;od;
```

Regular group $\mathsf{A} \le F_1$, $\mathsf{A} \cong H_{[96,13]}$, must be given through a permutation representation of degree 96. The one used here, deduced from Table 1, we give in Table 9 in GAP straightly readable form. Also, in this case,
`ds:=`[1,2,4,5,6,15,21,28,35,38,39,53,55,61,64,65,74,75,89,92].

Procedure returns a list `delta:`

[<identity ...>, f1^2*f2^3*f3*f4*f5, f1*f5, f1*f2^2*f3, f1^2*f2^2*f3*f5, f1^2*f2^3*f3*f4, f1^2*f5, f1*f2*f3*f5, f1*f2*f4*f5, f1^2*f2^3*f3*f5, f3*f4, f1^2*f3, f1*f2^2, f1*f4, f1*f3*f4, f1*f2^2*f5, f1^2*f3*f4, f2, f1^2*f2*f3*f4, f2^3*f3] of elements of group $\mathsf{H} \cong H_{[96,13]}$ corresponding to the set `ds`.

This result, on the basis of isomorphism in fact a difference set $\Delta^1_{[96,13]} \subseteq H_{[96,13]}$, is easily trasformed into terms of generators x,y,z,t,a of the group $H_{[96,13]}$. As it is accustomed to identify the subset Δ of a group with the group ring element $\Delta = \sum_{d\in\Delta} d$, in the group ring $ZH_{[96,13]}$ we finally have

$$\Delta^1_{[96,13]} = 1 + a^2x^3yzt + at + ax^2y + a^2x^2yt + a^2x^3yz + a^2t + axyt + axzt + a^2x^3yt + yz + a^2y + ax^2 + az + ayz + ax^2t + a^2yz + x + a^2xyz + x^3y.$$

TABLE 9. *Generators of the group* A, *GAP-cn :[96,13]*

Group([(1,4)(2,3)(5,6)(7,49)(8,93)(9,25)(10,81)(11,18)(12,58)(13,22)(14,89)(15,20)(16,72)(17,91)
(19,90)(21,86)(23,88)(24,63)(26,62)(27,64)(28,60)(29,43)(30,59)(31,41)(32,42)(33,56)(34,61)(35,54)
(36,65)(37,47)(38,79)(39,85)(40,68)(44,96)(45,76)(46,92)(48,78)(50,77)(51,70)(52,75)(53,66)(55,83)
(57,82)(67,84)(69,74)(71,94)(73,80)(87,95),
(1,52,22,10)(2,86,67,44)(3,34,31,6)(4,54,19,49)(5,38,64,58)(7,29,9,36)(8,16,28,90)(11,14,30,33)
(12,45,88,21)(13,92,72,25)(15,18,57,37)(17,20,73,70)(23,61,78,55)(24,27,84,83)(26,41,76,63)

(32,81,43,60)(35,42,46,39)(40,56,80,87)(47,94,74,95)(48,96,79,62)(50,53,82,68)(51,59,77,69)
(65,93,85,75)(66,89,91,71),
(1,35,90,7)(2,61,41,5)(3,21,84,96)(4,75,13,81)(6,79,27,12)(8,39,52,36)(9,22,46,16)(10,29,28,42)
(11,82,47,20)(14,17,94,53)(15,80,51,91)(18,89,59,56)(19,93,72,60)(23,86,58,76)(24,62,31,45)
(25,65,49,43)(26,78,44,38)(30,50,74,70)(32,92,85,54)(33,73,95,68)(34,48,83,88)(37,71,69,87)
(40,77,66,57)(55,63,64,67),
(1,22)(2,67)(3,31)(4,19)(5,64)(6,34)(7,9)(8,28)(10,52)(11,30)(12,88)(13,72)(14,33)(15,57)(16,90)
(17,73)(18,37)(20,70)(21,45)(23,78)(24,84)(25,92)(26,76)(27,83)(29,36)(32,43)(35,46)(38,58)(39,42)
(40,80)(41,63)(44,86)(47,74)(48,79)(49,54)(50,82)(51,77)(53,68)(55,61)(56,87)(59,69)(60,81)(62,96)
(65,85)(66,91)(71,89)(75,93)(94,95),
(1,90)(2,41)(3,84)(4,13)(5,61)(6,27)(7,35)(8,52)(9,46)(10,28)(11,47)(12,79)(14,94)(15,51)(16,22)
(17,53)(18,59)(19,72)(20,82)(21,96)(23,58)(24,31)(25,49)(26,44)(29,42)(30,74)(32,85)(33,95)(34,83)
(36,39)(37,69)(38,78)(40,66)(43,65)(45,62)(48,88)(50,70)(54,92)(55,64)(56,89)(57,77)(60,93)(63,67)
(68,73)(71,87)(75,81)(76,86)(80,91),
(1,86,73)(2,10,20)(3,15,81)(4,80,21)(5,47,42)(6,32,37)(7,33,23)(8,50,63)(9,14,78)(11,29,61)(12,87,54)
(13,91,96)(16,26,53)(17,22,44)(18,34,43)(19,40,45)(24,77,93)(25,48,89)(27,85,69)(28,82,41)(30,36,55)
(31,57,60)(35,95,58)(38,46,94)(39,64,74)(49,88,56)(51,75,84)(52,70,67)(59,83,65)(62,72,66)(68,90,76)
(71,92,79)])

In the same way we find difference sets in all the above listed groups.

$$\begin{aligned}
\Delta^1_{[96,41]} &= 1+a^2+zt+azt+a+a^2xz+a^2x^3y+x^2z+xy+a^2x^2+x^3yz+\\
&\quad a^2x^2y+a^2xyt+ayt+a^2xzt+x^3t+a^2y+t+ax^2yz+x^3zt,\\
\Delta^1_{[96,64]} &= 1+a^2x^2z+a^2x^2axaz+axa^2x^2z+axax+a^2xz+x^2axaz+\\
&\quad x^2axz+ax^2a^2x+xa^2z+xaxz+x^2axa+a^2xaxz+axa^2z+axax^2az+\\
&\quad axa^2x^2+ax^2ax+a^2xax^2+x^2a^2+ax^2z,\\
\Delta^1_{[96,70]} &= 1+a^2x+xaya^2+a^3xa+ay+a^4xaya^2+a^5xa^2+a^5xya+a^5ya+\\
&\quad axa+a^5xa+a^5xaya^2+a^2xa^2+a^2xa^2y+a^3xay+xa^2ya+\\
&\quad a^4xa^2ya^2+a^2xay+ya+a^3,\\
\Delta^1_{[96,71]} &= 1+ax^2az+a^2x^2ax+xax+xax^2a+xa^2z+a^2xz+ax^2a^2z+a^2xax+\\
&\quad axax^2z+axa^2z+axax^2+a^2xa^2+a^2xax^2a+a^2xa^2x^2+ax^2a^2xz+\\
&\quad ax^2a+axa^2xz+ax^2ax+ax^2axa,\\
\Delta^1_{[96,160]} &= 1+a^2xyzv+aytv+a^2xzt+xzv+xtv+xztv+xzt+y+a^2ztv+xyzt+\\
&\quad a^2xztv+a^2t+a^2yzt+a^2xyt+axztv+a^2yzv+axyv+xv+azv,\\
\Delta^1_{[96,167]} &= 1+a^2xzt+a^3x+a^4xax+a^3xaxt+a^3xazt+a^3xa+t+a^5xaxz+\\
&\quad axax+a^5xaxzt+axaxz+a^5x+a^4xa+a^5+a^2xaxz+a^2xt+a^2xat+\\
&\quad a^3xaxz+a^3zt,\\
\Delta^1_{[96,185]} &= 1+ax^2yaya+x^3yaya+a^2x^3yay+a^2x+a^2x^3+xyay+y^2+yay^2+\\
&\quad ay+ya+a^2yay^2+ax^3y^2+axy+a^2x^3yay^2+x^3ya+ax^2+ax^3yay+\\
&\quad x^3yay+xyay^2,\\
\Delta^1_{[96,186]} &= 1+a^2x^2+ax^3yay+axya^2y+ay+ax^2y+a^2x^2ya^2yay+ax^2yay+\\
&\quad x^3ya^2yay+ax^3ya^2yay+xya+a^2xya^2+a^2x+y+aya+aya^2+\\
&\quad aya^2y+x^3+a^2ya^2yay+a^2y,\\
\Delta^1_{[96,188]} &= 1+axax^2+axa^2xa+x^2a+a^2x^2ax+axa^2x+a^2x^2ay+axax^2y+\\
&\quad xa+axax^2ay+x^2a^2x+axaxy+a^2xay+axa^2xa^2+xax^2ay+\\
&\quad a^2xax^2ay+axa^2x^2y+ax^2ax+a^2x^2+a^2x,
\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,190]} &= 1 + ax^2a^2xy + ax^2a^2y + ax^4 + a^2xa^2y + x^3y + a^2y + axa^2+\\ &a^2xa^2x + a^2 + axa^2x^2y + xa^2xay + a + x^2ay + axa^2xa^2y + a^2x^3+\\ &xy + ax^3a^2y + a^2x^2 + y,\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,194]} &= 1 + a + a^2x^3yt + a^2x^3yat + a^2 + a^2x^2t + ax^2yat + y + yt+\\ &a^2x^2ya^2 + t + ax^2 + axy + ax^3 + xya^2t + x^3ya + a^2ya^2 + a^2x+\\ &aya + axya^2t,\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,195]} &= 1 + axayv + a^2xaxav + xv + xa^2y + axay + xa^2v + xav + a^2xax+\\ &axa + axv + a^2y + axa^2xyv + a^2xaxyv + axaxaxv + axa^2xy + axaxv+\\ &a^2xa + a^2xv + axyv,\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,196]} &= 1 + a^{10}x + a^3xy + a^7xy + a^{11}xy + a^5xy + a^{11}y + a^{10}xa^2 + a^{10}xa^2y+\\ &a^9xay + y + a^{10}y + a^5xa^2y + ax + a^{11}xa^2 + axa^2 + a^2xa^2 + a^2xay+\\ &a^5y + a^9,\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,197]} &= 1 + a^2 + a^2xaz^3 + a^2xz^3 + a^2xa^2z^3 + a^2xa^2z + az + xz^2 + a^2xazt+\\ &a^2xz^3t + axa^2z^3t + xa^2zt + xa^2z^3 + a^2xz^2t + xa^2z^2t + axa^2t + axaz^2+\\ &axa^2z^3 + az^3 + a^2xat,\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,226]} &= 1 + xt + xa^2 + xaxzt + xaz + xa^2xaxz + xa + axaxzt + a^2xa^2xz+\\ &axat + axa^2xat + axa + a^2z + xa^2xa + axa^2xt + a^2xa + x + xa^2zt+\\ &xa^2xax + a^2xazt,\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,227]} &= 1 + v + axaza^2v + axaza^2 + a^2xzv + xa^2zv + axazv + axa^2z+\\ &xa^2za + a^2xazv + a^2xa^2za^2 + xaza^2v + za + xa + a^2xa^2z + axa^2zav+\\ &axazav + axzv + a^2zv + axv,\end{aligned}$$

$$\begin{aligned}\Delta^1_{[96,228]} &= 1 + a^4 + a^3 + a^2 + a^5 + a^2xy + a^3xaxay + axa^2 + a^2xa + a^3xax+\\ &a^4xaxa + a^5xaxaxy + xa^2x + xa^2xy + a^2y + a^5xay + a^5xa^2y + x+\\ &xaxax + a^2xaxy.\end{aligned}$$

$\Delta^{\mathsf{i}}_{[96,\mathsf{j}]}$ denotes a difference set obtained from the symmetric design $D_{\mathsf{i}}, i = 1, \dots, 8$ in the group $H_{[96,\mathsf{j}]}$.

Without going into details of the procedure, now we give final results for the groups $F_{\mathsf{i}} = AutD_{\mathsf{i}}, i = 2, \dots, 8$.

$F_2 = AutD_2$ has four nonisomorphic subgroups acting regularly on D_2 and they are, up to isomorphism, the groups with the GAP-cns: [96,159], [96,160], [96,229], and [96,231]. Excluding priviously given [96,160], in terms of generators and relations they can be given as:

$$\begin{aligned}H_{[96,159]} = \quad &\langle x, y, z, a \mid x^4 = y^2 = z^2 = [x, y] = [y, z] = 1, x^{\mathsf{z}} = xy, a^3 = 1,\\ &a^{\mathsf{x}} = a^{-1}, [a, y] = [a, z] = 1 \quad \times \langle t \mid t^2 = 1\rangle,\end{aligned}$$

$$\begin{aligned}H_{[96,229]} = \quad &\langle x, y, z, t, a \mid x^2 = y^2 = z^2 = t^2 = [x, y] = [x, z] = [x, t] = [y, z] = 1,\\ &[y, t] = [z, t] = 1, a^6 = 1,\ x^{\mathsf{a}} = xz,\ y^{\mathsf{a}} = yt, z^{\mathsf{a}} = x, t^{\mathsf{a}} = y \quad ,\end{aligned}$$

$$H_{[96,231]} = \quad Z_2^5 \times Z_3 = \langle x, y, z, t, v\rangle \times \langle a \mid a^3 = 1\rangle.$$

The corresponding difference sets are:

$$\begin{aligned}\Delta^2_{[96,159]} &= 1 + a^2x^3zt + x^2yzt + a^2xy + a^2 + a^2x^2zt + xyzt + x^3y + x^2yt + a^2xyt+\\ &x^2z + a^2xz + a + ax^2y + ax^2 + ay + a^2x^2yt + x^3yz + a^2x^2yz + x^3yt,\end{aligned}$$

$$\begin{aligned}\Delta^2_{[96,160]} &= 1 + axztv + xt + axyv + a + ayzt + yzv + xyztv + axyzv + xy + axzv+\\ &yz + a^2 + a^2xyt + a^2zt + a^2xyz + xyzv + axzt + yv + axyzt,\end{aligned}$$

$$\Delta^2_{[96,229]} = 1 + a^4xa^2 + a^5xa + x + xa^2 + a^5xaya^2 + a^2y + axya + a^3 + a^3xy+$$

$$a^2xya + axya^2 + a^4x + a^4xy + axa^2ya + axaya^2 + a^3xa^2 + a^4ya + a^5xy + axa^2ya^2,$$

$$\Delta^2_{[96,231]} = 1 + xyz + xt + yzt + a^2 + a^2t + a^2z + a^2zt + v + xztv + xytv + yzv + a + axy + ay + ax + a^2v + a^2xzv + a^2ytv + a^2xyztv.$$

$F_3 = AutD_3$ has the following eight subgroups acting regularly on D_3: [96,70], [96,87], [96,144], [96,167], [96,227], [96,228], [96,229], and [96,230]. Excluding priviously given ones, in terms of generators and relations they can be given as:

$$H_{[96,87]} = \langle x, y, z \mid x^4 = y^2 = z^2 = [x,y] = [y,z] = 1,\ x^{\mathsf{z}} = xy\rangle \times \times\langle a, t \mid a^3 = t^2 = (at)^2 = 1\rangle,$$

$$H_{[96,144]} = \langle x, y, z, t, a \mid x^4 = y^2 = z^2 = [x,z] = [y,z] = t^2 = [t,y] = [t,z] = 1, x^{\mathsf{y}} = x^{-1}, x^{\mathsf{t}} = xz, a^3 = 1, a^{\mathsf{x}} = a^{-1}, a^{\mathsf{y}} = a^{-1}, [a,z] = [a,t] = 1\rangle,$$

$$H_{[96,230]} = Z_2^4 \times D_6 = \langle x, y, z, t\rangle \times \langle a, v \mid v^2 = a^3 = (va)^2 = 1\rangle.$$

The corresponding difference sets are:

$$\Delta^3_{[96,70]} = 1 + a^2xa + ya^2 + a^4y + a^2xaya^2 + a^3xay + a^4xy + a^4xya^2 + a^3xa^2 + a^5y + axaya + a^2xa^2y + axya^2 + a^2 + a^2y + a^5ya + xy + a^4ya + a^3y + axya,$$

$$\Delta^3_{[96,87]} = 1 + at + a + a^2 + t + a^2x^2y + a^2x^2 + x^2yz + yzt + axyz + x^3z + a^2y + azt + ax^2z + x^3yzt + axzt + ax^3yt + ax + x^3 + xyt,$$

$$\Delta^3_{[96,144]} = 1 + xy + a + a^2 + axy + a^2x^2 + a^2x^2z + zt + ax^3yt + ax^2yz + a^2yz + a^2z + x^3yzt + at + ax + a^2x^3 + a^2x^3zt + ayzt + a^2x^2yt + axt,$$

$$\Delta^3_{[96,167]} = 1 + a^2xaz + a^2 + a^4 + a^4xaz + a^4xz + a^3xaz + z + a^5z + a^4xaxzt + a^2xat + a^3xax + a^2xaxz + axaxz + axaxzt + a^5xat + a^5xazt + a^5xzt + a^2xaxt + a^2t,$$

$$\Delta^3_{[96,227]} = 1 + v + xaz + a^2xaza^2 + xazv + axaz + axa^2za^2 + z + axa^2zav + a^2xa^2v + a^2xa^2za^2v + a^2xz + axa^2zv + xza + a^2xaza + a^2xaz + a^2xa + a^2za^2v + xa^2zav + a^2xa^2z,$$

$$\Delta^3_{[96,228]} = 1 + a^5xaxy + a^3xax + xa^2x + xaxay + a^4xay + a^5y + a^3y + axaxy + xa^2xy + axaxax + xaxax + a^4xa^2y + axy + a^5xa^2xy + x + a^3xy + a^2y + a^3 + a^2xaxax,$$

$$\Delta^3_{[96,229]} = 1 + a^3 + a^2xa^2y + a^4xa^2ya^2 + axa^2ya^2 + a^2xy + xya^2 + xy + a^4xya + a^4xa + axa^2y + a^2y + a^5xa^2ya^2 + a^3ya + a^3xya + a^5xay + a^4xa^2 + a^4xay + xa^2ya + a^4xy,$$

$$\Delta^3_{[96,230]} = 1 + axyv + a + a^2 + xyv + a^2yzt + a^2xz + t + yztv + ztv + ayv + a^2xyt + axzv + axy + ayt + z + zt + xv + azv + axt.$$

Now we list GAP-cns of regular subgroups in the remaining full automorphism groups $F_{\mathsf{i}}, i = 4, \ldots, 8$.

In F_4 : [96,70], [96,167], [96,196], [96,197], [96,226], [96,227], [96,228], [96,229];

in F_5 : [96,227], [96,228], [96,229];

in F_6 : [96,167], [96,195], [96,227];

in F_7 : [96,13], [96,41], [96,167], [96,160], [96,227];

in F_8 : [96,64], [96,70], [96,71], [96,195], [96,190], [96,227].

As all these groups have already appeared in our considerations, we don't cite the additional 25 difference sets constructed in them from symmetric designs $D_{\mathsf{i}}, i = 4, \ldots, 8$.

Remark 3.1. *Fifty five difference sets distinguished in this section are inequivalent, being nonisomorphic or contained in nonisomorphic groups. Moreover, it turns out that, starting from symmetric designs $D_i, i = 1, \ldots, 8$, in some of the cited groups even greater number of inequivalent difference sets can be obtained. If the group $AutD_3 \cong F_i$ has more than one conjugacy class of subgroups isomorphic to $H_{[96,j]}$, difference sets constructed in the class representatives are not necessarily equivalent. For instance, F_1 has two conjugacy classes of the subgroups isomorphic to $H_{[96,194]}$ and difference sets constructed in the representatives of these classes have nonisomorphic groups of right multipliers (Theorem 1.3), thus being inequivalent. Similarly, starting from D_1 the existence of at least four inequivalent difference sets in the group $H_{[96,195]}$ can be proved. We won't go further into details regarding the number of inequivalent difference sets contained in particular group because in this paper we focus on obtaining difference sets in as many nonisomorphic groups as possible.*

Remark 3.2. *Among the full automorphism groups $F_i, i = 1, \ldots, 8$, only F_2 contains regular abelian subgroup, precisely $Z_2^5 \times Z_3$, i.e. [96,231]. This implies that difference sets $\Delta^i_{[96,j]}, i = 1, 3, 4, 5, 6, 7, 8$ are genuinely nonabelian.*

4. Surway of results

Out of 24 groups in which we've constructed difference sets in Section 3 one is abelian ([96,231]), while the group [96,230], having Z_2^4 as its center, admits the well known Dillon's difference set construction. Thus, the results obtained prove the existence of difference sets in 22 so far undecided nonabelian groups of order 96. Considering that Theorem 1.1 in this sense rules out the generalized dihedral groups [96,6], [96,81], [96,110], and [96,207], and that Dillon's construction works out for only one more group of order 96 (precisely the group [96,218]), we can summarize the known difference sets existence status for the groups of order 96 in the following tabular form.

GAP-cn	Existence	References	GAP-cn	Existence	References
$[96, 2]$	No	$Z_{32} \times Z_3$	$[96, 185]$	Yes	D_1
$[96, 6]$	No	Theorem 1.1	$[96, 186]$	Yes	D_1
$[96, 13]$	Yes	D_1, D_2, D_7	$[96, 188]$	Yes	D_1
$[96, 41]$	Yes	D_1, D_2, D_7	$[96, 190]$	Yes	D_1, D_8
$[96, 46]$	No	$Z_4 \times Z_8 \times Z_3$	$[96, 194]$	Yes	D_1
$[96, 59]$	No	$Z_2 \times Z_{16} \times Z_3$	$[96, 195]$	Yes	D_1, D_6, D_8
$[96, 64]$	Yes	D_1, D_8	$[96, 196]$	Yes	D_1, D_4
$[96, 70]$	Yes	D_1, D_3, D_4, D_8	$[96, 197]$	Yes	D_1, D_4
$[96, 71]$	Yes	D_1, D_8	$[96, 207]$	No	Theorem 1.1
$[96, 81]$	No	Theorem 1.1	$[96, 218]$	Yes	Dillon's con.
$[96, 87]$	Yes	D_3	$[96, 220]$	Yes	$Z_2^3 \times Z_4 \times Z_3$
$[96, 110]$	No	Theorem 1.1	$[96, 226]$	Yes	$D_1, D_4,$
$[96, 144]$	Yes	D_3	$[96, 227]$	Yes	$D_1, D_3, D_4, D_5,$
$[96, 159]$	Yes	D_2			D_6, D_7, D_8
$[96, 160]$	Yes	D_1, D_2, D_7	$[96, 228]$	Yes	D_1, D_3, D_4, D_5
$[96, 161]$	Yes	$Z_2 \times Z_4^2 \times Z_3$	$[96, 229]$	Yes	D_2, D_3, D_4, D_5
$[96, 167]$	Yes	D_1, D_3, D_4, D_6, D_7	$[96, 230]$	Yes	D_3,Dillon's con.
$[96, 176]$	No	$Z_2^2 \times Z_8 \times Z_3$	$[96, 231]$	Yes	$Z_2^5 \times Z_3, D_2$

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