

Is the Stationary Current Distribution in a Linear Planar Electric Network Determined by the Principle of Maximum Entropy Production?*

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The standard derivation of Kirchhoff's voltage law is based on the assumption that the first law of thermodynamics (sum of products of applied EMF's and corresponding currents = heat dissipation) is satisfied within each loop. In contrast, we start from the fact that first law of thermodynamics applies globally to the electric circuit as a whole. It is then shown that Kirchhoff's voltage law may be derived as an extremum of the system entropy production constrained by the first law applied globally, and that the extremum is a maximum. Thus the stationary distribution of currents in a linear electric circuit is governed by the principle of maximum entropy production.

Keywords
maximum entropy production
Kirchhoff's voltage law

INTRODUCTION

Following Jaynes' information theory formalism of non-equilibrium statistical mechanics^{1,2} Dewar³ has put forward a statistical mechanical predictive framework for systems in non-equilibrium stationary states. In this information theoretic approach even if the system is arbitrary far from equilibrium the temperature, non-equilibrium entropy density and entropy production are physically well defined quantities. It is shown in Ref. 3 that maximizing the path information entropy according to Jaynes' procedure of maximizing information entropy is equivalent to maximizing the mean entropy production. In this article we investigate the existence of a differen-

tial extremum principle governing energy conversion in the planar electric network consisting of linear resistors and constant electromotive forces (EMF-s). We shall show that observed stationary current distribution is accompanied by maximum entropy production.

NECESSARY AND SUFFICIENT CONDITIONS FOR MAXIMUM ENTROPY PRODUCTION

Mesh Currents as Thermodynamic Variables of the Planar Electric Network

Stationary currents in a planar electric network consisting of linear resistors and constant EMF-s obey Ohm's

* Dedicated to Dr. Edward C. Kirby on the occasion of his 70th birthday.

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law, Kirchoff's current law (KCL) and Kirchoff's voltage law (KVL). In electric circuit analysis there is an elegant method based on the notion of mesh current that enables one to calculate all unknown currents in a linear planar network from the system of linear algebraic equations deduced solely from KVL.

The basic notions in this analysis are a node, loop and simple loop (mesh). The node is the common point of three or more branches. A loop is a closed path along the elements in the circuit passing through no node or element more than once. A mesh is a loop that contains no internal loop (see Figure 1). In other words the meshes are the simplest possible loops analogous to the meshes of a fish net.⁴ Further, one associates a current J_k ($k = 1, 2, \dots, n$) with each mesh (mesh current) (Figure 2). Mesh currents need not be the real physical currents (currents flowing through branches). The current flowing through a branch that is common to two neighboring meshes is the algebraic sum of corresponding mesh currents (current J_a in Figure 2), while the current flowing through the branch belonging only to one mesh is equal to the mesh current associated with that loop (currents J_b and J_c in Figure 2). Thus the concept of mesh currents ensures KCL (= charge conservation) to be implicitly taken into account. The number of linear algebraic equations deduced solely from KVL is equal to the number of unknown mesh currents J_k ($k = 1, 2, \dots, n$).

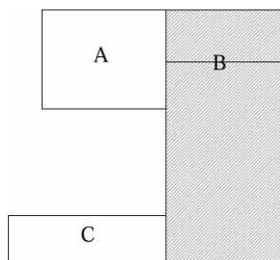


Figure 1. B is a loop, A and C are meshes.

In the following we consider in turn an electric network with two meshes and an arbitrary complex network with more than two meshes. When applied to an electric network consisting of two meshes (Figure 2) KVL leads to two linear equations for the two unknown mesh currents J_1 and J_2 :

$$\begin{aligned} E_{11} + E_{12} &= J_1 R_{11} + J_1 R_{12} - J_2 R_{21} \\ E_{22} + E_{21} &= J_2 R_{22} + J_2 R_{21} - J_1 R_{12}. \end{aligned} \quad (1)$$

It is evident that in the case of an electric network with n meshes KVL demands that the set of n mesh currents J_1, J_2, \dots, J_n satisfy the system of n algebraic linear equations⁶

$$\sum_l E_{kl} = R_{kk} J_k + \sum_l R_{kl} (J_k - J_l) \quad (2)$$

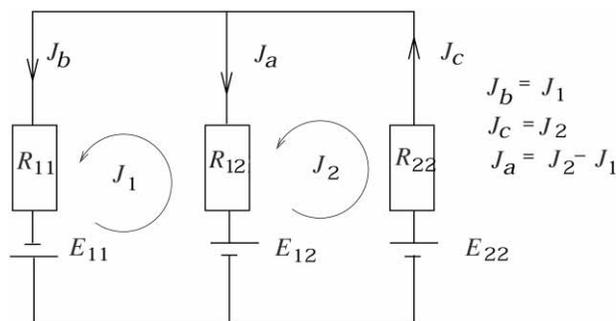


Figure 2. Mesh currents and currents flowing through branches.

Here E_{kl} and R_{kl} are equivalent EMF-s and equivalent resistances, respectively. By definition an equivalent EMF E_{kl} is the algebraic sum of EMF's within a branch, *i.e.* summing EMF's within a branch we have to care about their polarization. An equivalent resistance R_{kl} is just sum of resistances of resistors, connected in series, within a given branch of the network, *i.e.* it is equal to the equivalent resistance of the resistors connected in series.

In (1) and (2) different indices $k \neq l$ denote the elements (EMF-s and resistors) in the branch common to the two neighboring meshes, while E_{kk} and R_{kk} are the elements in the outer branch (the branch that belongs to only one mesh). We also have

$$E_{kl} = -E_{lk} \text{ for } k \neq l, \quad (3)$$

and

$$R_{kl} = R_{lk}. \quad (4)$$

The First Law of Thermodynamics in Terms of Mesh Currents

A linear planar electric network consists of two parts: an active part consisting of all EMF-s and a passive part comprising all linear resistors. When stationary currents flow in such a network the passive part converts the energy of all EMFs into Joule heat that is given off to the surroundings (a reservoir of constant temperature T). Then the stationary state of the linear planar network as the thermodynamic system at absolute temperature T is completely determined by the set of n mesh currents J_k ($k = 1, 2, \dots, n$). During the conversion of electric energy into heat there is no change of the internal energy of the passive part (resistors). Therefore, according to the first law of thermodynamics the rate at which energy is being delivered to the passive part, dW/dt , should be equal to the rate of energy conversion into heat, dQ/dt , *i.e.*

$$\frac{dW}{dt} = \frac{dQ}{dt}. \quad (5)$$

When stationary current J flows through a conductor with resistance R the rate of heat production is given by Joule's experimental law⁵

$$\frac{dQ}{dt} = RJ^2. \quad (6)$$

By making use of relations (3), (4) and (6) we get for the network shown in Figure 2:

$$\begin{aligned} \frac{dW}{dt} &= E_{11}J_1 + E_{22}J_2 + E_{12}(J_1 - J_2), \\ \frac{dQ}{dt} &= R_{11}J_1^2 + R_{22}J_2^2 + R_{12}(J_1 - J_2)^2. \end{aligned} \quad (7)$$

The relations (7) are easily generalized for a linear planar electric network with n meshes. Then by means of Eq. (5) we get:

$$\begin{aligned} \sum_k E_{kk}J_k + \frac{1}{2} \sum_{k,l} E_{kl}(J_k - J_l) = \\ \sum_k R_{kk}J_k^2 + \frac{1}{2} \sum_{k,l} R_{kl}(J_k - J_l)^2. \end{aligned} \quad (8)$$

Here the factor 1/2 appears due to double counting in the case of the branches common to the neighboring meshes.

We note that KVL assumes that the first law of thermodynamics holds for each loop separately. From the thermodynamical point of view this assumption is not at all obvious a priori.

Entropy Production in Terms of Mesh Currents

The conversion of electrical energy into Joule heat in a resistor is an irreversible process. If current J flows through a resistor with resistance R , then the rate of this process is determined by the entropy production:

$$\frac{d_i S}{dt} = \frac{RJ^2}{T} \quad (9)$$

where T is the absolute temperature of the resistor. Note that Eq. (9) incorporates Joule's law.

Since entropy production is an additive quantity it is evident that for the network shown in Figure 2 the overall entropy production is given by

$$\frac{d_i S}{dt} = \frac{1}{T} [R_{11}J_1^2 + R_{22}J_2^2 + R_{12}(J_1 - J_2)^2] \quad (10)$$

where an assumption is made that all the resistors in the network are at the same temperature T . With the same assumption the straightforward generalization of (10) to the network with n meshes yields

$$\frac{d_i S}{dt} = \frac{1}{T} \left[\sum_k R_{kk}J_k^2 + \frac{1}{2} \sum_{k,l} R_{kl}(J_k - J_l)^2 \right]. \quad (11)$$

Being defined in n -dimensional linear space the entropy production (11) is a differentiable function of n independent mesh currents.

Maximum Entropy Production and Kirchhoff's Voltage Law

For the given equivalent EMF-s and equivalent resistances in the network with n meshes, where $n \geq 2$, there exist infinitely many sets of n stationary mesh currents determining possible stationary states satisfying the global energy conservation law of Eq. (8). However only the set of n stationary mesh currents obtained as the unique solution of the system of n algebraic Eqs. (2) determines the actual stationary state which occurs in nature. It will be shown that a differential extremum principle obeyed by entropy production (11) is a selection principle from which one derives the system of Eqs. (2).

The actual stationary currents flowing in a linear planar electric network are obtained by the procedure of seeking for the extremum of entropy subject to Eq. (8). This is a problem of conditional extremum for entropy production (11) in which the law of global energy conservation (8) plays the role of constraint. In other words the system entropy production (11) is constrained by

$$\begin{aligned} \psi = \frac{1}{T} \left[\sum_k E_{kk}J_k + \frac{1}{2} \sum_{k,l} E_{kl}(J_k - J_l) - \right. \\ \left. - \sum_k R_{kk}J_k^2 - \frac{1}{2} \sum_{k,l} R_{kl}(J_k - J_l)^2 \right] = 0. \end{aligned} \quad (12)$$

Then according to the standard procedure the function F ,^{6,7}

$$F = \frac{d_i S}{dt} + \lambda \psi \quad (13)$$

has the extremum at the point of n -dimensional linear space (J_1, J_2, \dots, J_n) , which is the conditional extremum point of entropy production $d_i S/dt$. The Lagrange multiplier λ and the conditional extremum point of entropy production are obtained from the conditions

$$T \frac{\partial F}{\partial J_k} = 0, \quad (14)$$

$$T\psi = 0. \quad (15)$$

The conditions (14) yield the system of n linear algebraic equations

$$T \frac{\partial F}{\partial J_k} = 2(l - \lambda) \left[R_{kk}J_k + \sum_l R_{kl}(J_k - J_l) \right] + \lambda \sum_l E_{kl} = 0. \quad (16)$$

In the case $n = 2$ the system reads

$$R_{11}J_1 + R_{12}(J_1 - J_2) = \frac{-\lambda}{2 - 2\lambda} (E_{11} + E_{12})$$

$$R_{21}(J_2 - J_1) + R_{22}J_2 = \frac{-\lambda}{2 - 2\lambda} (E_{21} + E_{22}). \quad (17)$$

Multiplication of first and second Eq. in (17) by J_1 and J_2 respectively, and summing them yields

$$R_{11}J_1^2 + R_{22}J_2^2 + R_{12}J_1^2 + R_{21}J_2^2 - R_{12}J_1J_2 - R_{21}J_2J_1 = \frac{-\lambda}{2-2\lambda}(E_{11}J_1 + E_{22}J_2 + E_{12}J_1 + E_{21}J_2) \quad (18)$$

Comparison of condition (15) in the case $n = 2$ with Eq. (18) gives $\lambda = 2$. Due to (4) the system (17), for $\lambda = 2$, is identical to the system (1) that was derived solely from KVL.

By straightforward calculation it can be shown⁶ that for an arbitrary number of meshes $\lambda = 2$. This means that among all functions $F(\lambda)$ only $F(\lambda = 2)$ has a possible point of extremum uniquely determined by KVL [compare Eq.(16) and Eq.(2)].

Sufficient Condition for the Maximum of the Entropy Production

We have shown (Ref. 6 and this paper) that KVL (2) follows as the necessary condition for the extremum value of entropy production to exist.

Due to the relations (11) and (12) the second differential of function $F(\lambda = 2)$ defined by expression (13) could be written in the form:

$$d^2F = \frac{-2}{T} \left[\sum_k R_{kk} dJ_k^2 + \sum_{k,l} R_{kl} dJ_k^2 - \sum_{k,l} R_{kl} dJ_k dJ_l \right]. \quad (19)$$

Taking into account symmetry property (4) one obtains

$$d^2F = \frac{-2}{T} \left[\sum_k R_{kk} dJ_k^2 + \frac{1}{2} \sum_{k,l} R_{kl} (dJ_k - dJ_l)^2 \right] < 0. \quad (20)$$

We can conclude that if stationary currents satisfy the system of equations (2) the Lagrangian function (13) with $\lambda = 2$ reaches its maximum. This implies that entropy production (11) reaches its strict conditional maximum when stationary currents flow in a linear planar electric network with n meshes ($n \geq 2$). In other words the distribution of stationary currents in a linear planar electric network, obeying Joule's law, KCL, KVL and the law of energy conservation (12), is governed by the principle of maximum entropy production.

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REFERENCES

1. E. T. Jaynes, *Phys. Rev.* **106** (1957) 620–630.
2. E. T. Jaynes, *Phys. Rev.* **108** (1957) 171–190.
3. R. Dewar, *J. Phys. A. Math. Gen.* **36** (2003) 631–641.
4. L. Weinberg, *Network Analysis and Synthesis*, McGraw-Hill Book Company, Inc., New York, 1962, p. 136.
5. *McGraw-Hill Encyclopedia of Science & Technology*, McGraw-Hill Book Company, Inc., New York, 1987, vol. 9, pp. 470–471.
6. P. Županović, D. Juretić, and S. Botrić, *Phys. Rev. E* **70** (2004) 056108.
7. M. I. Krasnov, G. I. Makarenko, and A. I. Kiselev, *Problems and Exercises in the Calculus of Variation*, Mir Publishers, Moscow, 1975, pp. 19–22.

SAŽETAK

Je li razdioba stacionarnih struja u linearnim planarnim električnim mrežama određena principom maksimalne proizvodnje entropije?

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Uobičajeni izvod Kirchhoffovoga zakona za električnu petlju temelji se na pretpostavci da prvi zakon termodinamike (zbroy umnožaka narinutih elektromotornih sila i odgovarajućih struja = proizvedenoj toplini) vrijedi za svaku petlju. U ovom radu Kirchhoffov zakon za električnu petlju izvodi se pomoću principa maksimalne proizvodnje entropije, uz uvjet da prvi zakon termodinamike vrijedi za čitavu linearnu mrežu. Na taj način pokazuje se da se razdioba stacionarnih struja podvrgava principu maksimalne proizvodnje entropije.