Matematička logika i izračunljivost

Predavanje br. 8 na FER–u akad. god. 2009/2010

Mladen Vuković
vukovic@math.hr

PMF–Matematički odjel
Sveučilište u Zagrebu

06. studeni 2009.

Sadržaj predavanja:

Glavni test za logiku prvog reda

Uvod

Sada razmatramo jedan postupak koji može poslužiti prilikom rješavanja sljedećih problema:

- Ispitivanje **valjanosti** formule;
- Ispitivanje **ispunjivosti** formule;
- Ispitivanje **oborivosti** formule;
- Određivanje je li neka formula **logička posljedica** zadanog konačnog skupa formula;
- Određivanje jesu li zadane dvije formule **logički ekvivalentne**.

Uvod

Sada razmatramo jedan postupak koji može poslužiti prilikom rješavanja sljedećih problema:

- Ispitivanje **valjanosti** formule;
- Ispitivanje **ispunjivosti** formule;
- Ispitivanje **oborivosti** formule;
- Određivanje je li neka formula **logička posljedica** zadanog konačnog skupa formula;
- Određivanje jesu li zadane dvije formule **logički ekvivalentne**.

Uvod

Sada razmatramo jedan postupak koji može poslužiti prilikom rješavanja sljedećih problema:

- Ispitivanje **valjanosti** formule;
- Ispitivanje **ispunjivosti** formule;
- Ispitivanje **oborivosti** formule;
- Određivanje je li neka formula **logička posljedica** zadanog konačnog skupa formula;
- Određivanje jesu li zadane dvije formule **logički ekvivalentne**.

Uvod

Sada razmatramo jedan postupak koji može poslužiti prilikom rješavanja sljedećih problema:

- Ispitivanje **valjanosti** formule;
- Ispitivanje **ispunjivosti** formule;
- Ispitivanje **oborivosti** formule;
- Određivanje je li neka formula **logička posljedica** zadanog konačnog skupa formula;
- Određivanje jesu li zadane dvije formule **logički ekvivalentne**.

Uvod

Sada razmatramo jedan postupak koji može poslužiti prilikom rješavanja sljedećih problema:

- Ispitivanje **valjanosti** formule;
- Ispitivanje **ispunjivosti** formule;
- Ispitivanje **oborivosti** formule;
- Određivanje je li neka formula **logička posljedica** zadanog konačnog skupa formula;
- Određivanje jesu li zadane dvije formule **logički ekvivalentne**.

Kod logike sudova glavni test uvijek završava u konačno mnogo koraka i korektno odgovaraju na postavljeno pitanje.

Kod logike prvog reda to više nije slučaj.

Postoje oborive formule logike sudova za koju je svaka konačna struktura model.

To znači da će ponekad biti nemoguće odrediti (beskonačnu) strukturu u konačno mnogo koraka.

U ovoj točki promatramo samo zatvorene formule koje ne sadrže konstantske i funkcijske simbole.

Vidjet ćemo da je i taj smanjeni alfabet dovoljan kako bi se naglasila sva složenost problema ispitivanja valjanosti u logici prvog reda.

Pravila glavnog testa

Sada prvo navodimo pravila glavnog testa za propozicionalne veznike.

$(\neg)$

$$\begin{array}{l} \neg B \quad \textcircled{\top} \\ B \ \bot \end{array} \qquad \begin{array}{l} \neg B \quad \textcircled{\bot} \\ B \ \top \end{array}$$

$(\wedge)$

$$\begin{array}{l} B \wedge C \quad \textcircled{\top} \\ B \ \top \\ C \ \top \end{array} \qquad \begin{array}{c} B \wedge C \quad \textcircled{\bot} \\ / \quad \backslash \\ B \ \bot \quad C \ \bot \end{array}$$

$(\vee)$

$$\begin{array}{c} B \vee C \quad \textcircled{\top} \\ / \quad \backslash \\ B \ \top \quad C \ \top \end{array} \qquad \begin{array}{l} B \vee C \quad \textcircled{\bot} \\ B \ \bot \\ C \ \bot \end{array}$$

$(\rightarrow)$

$$\begin{array}{ccc} B \rightarrow C & & \top \\ / & \backslash & \\ B\ \bot & C\ \top & \end{array}$$

$$\begin{array}{cc} B \rightarrow C & \bot \\ B\ \top & \\ C\ \bot & \end{array}$$

$(\leftrightarrow)$

$$\begin{array}{ccc} B \leftrightarrow C & & \top \\ / & \backslash & \\ B\ \top & B\ \bot & \\ C\ \top & C\ \bot & \end{array}$$

$$\begin{array}{ccc} B \leftrightarrow C & & \bot \\ / & \backslash & \\ B\ \top & B\ \bot & \\ C\ \bot & C\ \top & \end{array}$$

Preostalo je napisati pravila glavnog testa za kvantifikatore.

No, opišimo prvo što zapravo znači ispitati valjanost neke formule F pomoću glavnog testa.

Ako ispitujemo je li neka formula F valjana tada je prvi redak testa oblika $F \ \bot$.

To znači da mi pokušavamo odrediti postoji li struktura koja nije model za formulu F.

Iz definicije strukture slijedi da moramo odrediti nosač $|\mathfrak{M}|$ i preslikavanje φ.

Opišimo prvo na koji način određujemo nosač $|\mathfrak{M}|$, tj. navedimo u kojim koracima ispitivanja **"punimo" nosač** s novim elementima.

Postoje dva takva osnovna oblika.

To su: $\forall x G(x)\ \bot$ **i** $\exists x G(x)\ \top$.

Za svaki od ta dva navedena oblika moramo u $|\mathfrak{M}|$ dodati novi element, jer npr. istinitost formule $\exists x G(x)$ znači da postoji element u $|\mathfrak{M}|$ koji je "svjedok" istinitosti.

Nakon analize retka oblika $\exists x G(x)\ \top$, prvo zaokružujemo znak $\top$, tj. pišemo $\textcircled{\top}$, te u taj redak dopisujemo $(..a..)$.

Element a mora biti novi, tj. ne smije biti uveden u nekom prethodnom koraku.

Oznaka $(..a..)$ nam sugerira da smo element a upravo uveli u tom koraku.

Zatim, u (nekom) sljedećem retku pišemo $G(a)$ $\top$.

Analogno postupamo za retke oblika $\forall x G(x)\ \bot$.

Time smo opisali dva oblika pravila za kvantifikatore.

Važno je istaknuti da u ovoj točki znakove $a, b, c \ldots, a_1, a_2, \ldots$ smatramo elementima alfabeta, tj. to su konstantski simboli u jednu ruku.

U drugu ruku te oznake koristimo za elemente nosača interpretacije.

Sada opisujemo pravila za slučajeve $\forall x G(x)$ $\top$ i $\exists x G(x)$ $\bot$.

Što zapravo znači istinitost formule oblika $\forall x G(x)$ za neku interpretaciju?

Po definiciji to znači istinitost formule $G(x)$ za svaku valuaciju.

Malo neprecizno, ali kraće zapisano, to zapravo znači istinitost formule $G(m)$, za svaki element $m \in |\mathfrak{M}|$.

Dakle, za svaki uvedeni element a u nosaču mi moramo ispitati istinitost formule $G(a)$.

Posebno to znači da ispitujemo za elemente koji su uvedeni prije i poslije retka $\forall x G(x)$ $\top$.

To pak povlači da redak oblika $\forall x G(x) \top$ možda nikad neće biti do kraja analiziran, jer se moguće poslije njega uvodi novi element u nosač.

Analizu retka oblika $\forall x G(x) \top$ u odnosu na element a označavamo sa $\forall x G(x) \top @a$.

Zatim, u (nekom) sljedećem retku pišemo $G(a) \top$.

Analogno postupamo za retke oblika $\exists x G(x)\ \bot$.

Sada navodimo pravila za kvantifikatore.

Oznaka ($\uparrow\ a \downarrow\ a$) nam sugerira da novo uvedeni element a moramo dopisati kod svih redaka oblika $\forall x G(x)\ \top$ i $\exists x G(x)\ \bot$ koji su bili prije i koji će se pojaviti kasnije.

$(\forall)$ $\quad \forall xB \ \top \ @, \ldots$

$B(a) \ \top$

$\forall xB \ \bigcirc{\scriptstyle \perp} \ (..a..) \ (\uparrow \ a \downarrow \ a)$

$B(a) \ \perp$

$(\exists)$ $\quad \exists xB \ \bigcirc{\scriptstyle \top} \ (..a..) \ (\uparrow \ a \downarrow \ a)$

$B(a) \ \top$

$\exists xB \ \perp \ @, \ldots$

$B(a) \ \perp$

Primjeri

Primjer 1

Ispitajmo valjanost formule $(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x))$

Na sljedećoj slici dano je jedno stablo glavnog testa za zadanu formulu.

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \bot$$

$$\forall x A(x) \to \exists x B(x) \quad \top$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \bot$$

$$\forall x A(x) \quad \bot \quad (..a..)$$

$$A(a) \quad \bot$$

$$(*) \quad \exists x (A(x) \to B(x)) \quad \bot \quad @a$$

$$A(a) \to B(a) \quad \bot$$

$$A(a) \quad \top$$

$$B(a) \quad \bot$$

$$X$$

$$\exists x B(x) \quad \top \quad (..b..)$$

$$B(b) \quad \top$$

$$(*) \quad \exists x (A(x) \to B(x)) \quad \bot \quad @b$$

$$A(b) \to B(b) \quad \bot$$

$$A(b) \quad \top$$

$$B(b) \quad \bot$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \bot$$

$$\forall x A(x) \to \exists x B(x) \quad \top$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \bot$$

$$\forall x A(x) \quad \bot \quad (..a..)$$

$$A(a) \quad \bot$$

$$(*) \quad \exists x (A(x) \to B(x)) \quad \bot \quad @$$

$$A(a) \to B(a) \quad \bot$$

$$A(a) \quad \top$$

$$B(a) \quad \bot$$

$$X$$

$$\exists x B(x) \quad \top \quad (..b..)$$

$$B(b) \quad \top$$

$$(*) \quad \exists x (A(x) \to B(x)) \quad \bot \quad @$$

$$A(b) \to B(b) \quad \bot$$

$$A(b) \quad \top$$

$$B(b) \quad \bot$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\bot}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \bot$$

$$\forall x A(x) \; \textcircled{\bot} \quad (..a..) \qquad\qquad \exists x B(x) \; \textcircled{\top} \quad (..b..)$$

$$A(a) \; \textcircled{\bot}$$

$$(*) \quad \exists x (A(x) \to B(x)) \; \bot \quad ⓐ$$

$$A(a) \to B(a) \; \textcircled{\bot}$$

$$A(a) \; \textcircled{\top}$$

$$B(a) \; \textcircled{\bot}$$

$$X$$

$$B(b) \; \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \; \bot \quad ⓑ$$

$$A(b) \to B(b) \; \textcircled{\bot}$$

$$A(b) \; \textcircled{\top}$$

$$B(b) \; \textcircled{\bot}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\perp} \qquad\qquad B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \ @a \qquad\qquad (*) \quad \exists x (A(x) \to B(x)) \ \perp \ @b$$

$$A(a) \to B(a) \textcircled{\perp} \qquad\qquad A(b) \to B(b) \textcircled{\perp}$$

$$A(a) \textcircled{\top} \qquad\qquad A(b) \textcircled{\top}$$

$$B(a) \textcircled{\perp} \qquad\qquad B(b) \textcircled{\perp}$$

$$X \qquad\qquad X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$\exists x B(x) \textcircled{\top} \quad (..b..)$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\scriptsize \perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\scriptsize \top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\scriptsize \perp} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\scriptsize \top} \quad (..b..)$$

$$A(a) \textcircled{\scriptsize \perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\scriptsize \perp}$$

$$A(a) \textcircled{\scriptsize \top}$$

$$B(a) \textcircled{\scriptsize \perp}$$

$$X$$

$$B(b) \textcircled{\scriptsize \top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\scriptsize \perp}$$

$$A(b) \textcircled{\scriptsize \top}$$

$$B(b) \textcircled{\scriptsize \perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \ \textcircled{\top}$$

$$B(a) \ \perp$$

$$X$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓐ \qquad\qquad (*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓑ$$

$$A(a) \to B(a) \textcircled{\perp} \qquad\qquad A(b) \to B(b) \textcircled{\perp}$$

$$A(a) \textcircled{\top} \qquad\qquad A(b) \textcircled{\top}$$

$$B(a) \ \perp \qquad\qquad B(b) \textcircled{\perp}$$

$$X \qquad\qquad X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$\exists x B(x) \quad \textcircled{\top} \quad (..b..)$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..) \qquad\qquad \exists x B(x) \quad \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓐ \qquad\qquad (*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓑ$$

$$A(a) \to B(a) \textcircled{\perp} \qquad\qquad A(b) \to B(b) \textcircled{\perp}$$

$$A(a) \textcircled{\top} \qquad\qquad A(b) \textcircled{\top}$$

$$B(a) \textcircled{\perp} \qquad\qquad B(b) \textcircled{\perp}$$

$$X \qquad\qquad X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{a}$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$\exists x B(x) \quad \textcircled{\top} \quad (..b..)$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{b}$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\bot}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \bot$$

$$\forall x A(x) \textcircled{\bot} \quad (..a..) \qquad\qquad \exists x B(x) \ \textcircled{\top} \quad (..b..)$$

$$A(a) \textcircled{\bot}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \bot \quad \textcircled{a} \qquad\qquad B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \ \bot \quad \textcircled{b}$$

$$A(a) \to B(a) \textcircled{\bot} \qquad\qquad A(b) \to B(b) \ \bot$$

$$A(a) \textcircled{\top} \qquad\qquad A(b) \textcircled{\top}$$

$$B(a) \textcircled{\bot} \qquad\qquad B(b) \textcircled{\bot}$$

$$X \qquad\qquad X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{a}$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$\exists x B(x) \quad \textcircled{\top} \quad (..b..)$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{b}$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \top$$

$$B(b) \perp$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓐ$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$\exists x B(x) \quad \textcircled{\top} \quad (..b..)$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad ⓑ$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \perp$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{a}$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$\exists x B(x) \quad \textcircled{\top} \quad (..b..)$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{b}$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

$$(\forall x A(x) \to \exists x B(x)) \to \exists x (A(x) \to B(x)) \quad \textcircled{\perp}$$

$$\forall x A(x) \to \exists x B(x) \quad \textcircled{\top}$$

$$(*) \qquad \exists x (A(x) \to B(x)) \quad \perp$$

$$\forall x A(x) \textcircled{\perp} \quad (..a..)$$

$$A(a) \textcircled{\perp}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{a}$$

$$A(a) \to B(a) \textcircled{\perp}$$

$$A(a) \textcircled{\top}$$

$$B(a) \textcircled{\perp}$$

$$X$$

$$\exists x B(x) \quad \textcircled{\top} \quad (..b..)$$

$$B(b) \textcircled{\top}$$

$$(*) \quad \exists x (A(x) \to B(x)) \perp \quad \textcircled{b}$$

$$A(b) \to B(b) \textcircled{\perp}$$

$$A(b) \textcircled{\top}$$

$$B(b) \textcircled{\perp}$$

$$X$$

Sljedećim primjerom želimo istaknuti nužnost uvođenja novih elemenata u nosač prilikom analize formula oblika $\forall x G(x)\ \bot$ i $\exists x G(x)\ \top$.

Primjer 2

Pomoću glavnog testa ispitajmo vrijedi li

$$\forall x \exists y P(x,y) \to \forall x Q(x) \models \forall x \forall y (P(x,y) \to Q(x)).$$

$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$

$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$

$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$

$P(a,b) \to Q(a)$ $\bot$

$P(a,b)$ $\top$

$Q(a)$ $\bot$

$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$ $\quad$ $\forall x Q(x)$ $\top$ @ b

$\exists y P(c,y)$ $\bot$ @ a b c $\quad$ $Q(a)$ $\top$

$P(c,a)$ $\bot$ $\quad$ X

$P(c,b)$ $\bot$

$P(c,c)$ $\bot$

$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$

$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$

$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$

$P(a,b) \to Q(a)$ $\bot$

$P(a,b)$ $\top$

$Q(a)$ $\bot$

$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$ $\quad$ $\forall x Q(x)$ $\top$ @ a b

$\exists y P(c,y)$ $\bot$ @ a b c $\quad$ $Q(a)$ $\top$

$P(c,a)$ $\bot$ $\quad$ X

$P(c,b)$ $\bot$

$P(c,c)$ $\bot$

$$\forall x \exists y P(x,y) \to \forall x Q(x) \quad \top$$

$$\forall x \forall y (P(x,y) \to Q(x)) \quad \bot \quad (..a..)$$

$$\forall y (P(a,y) \to Q(a)) \quad \bot \quad (..b..)$$

$$P(a,b) \to Q(a) \quad \bot$$

$$P(a,b) \quad \top$$

$$Q(a) \quad \bot$$

$$\forall x \exists y P(x,y) \; \bot \; (..c..) \qquad \forall x Q(x) \; \top \; @ \; b$$

$$\exists y P(c,y) \; \bot \; @ \; b \; c \qquad Q(a) \; \top$$

$$P(c,a) \; \bot \qquad X$$

$$P(c,b) \; \bot$$

$$P(c,c) \; \bot$$

$$\forall x \exists y P(x,y) \to \forall x Q(x) \quad \top$$

$$\forall x \forall y (P(x,y) \to Q(x)) \quad \bot \quad (..a..)$$

$$\forall y (P(a,y) \to Q(a)) \quad \bot \quad (..b..)$$

$$P(a,b) \to Q(a) \quad \bot$$

$$P(a,b) \quad \top$$

$$Q(a) \quad \bot$$

$$\forall x \exists y P(x,y) \; \bot \; (..c..) \qquad \forall x Q(x) \; \top \; a \; b$$

$$\exists y P(c,y) \; \bot \; a \; b \; c \qquad Q(a) \; \top$$

$$P(c,a) \; \bot \qquad X$$

$$P(c,b) \; \bot$$

$$P(c,c) \; \bot$$

$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$

$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$

$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$

$P(a,b) \to Q(a)$ $\bot$

$P(a,b)$ $\top$

$Q(a)$ $\bot$

$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$

$\exists y P(c,y)$ $\bot$ @a @b @c

$P(c,a)$ $\bot$

$P(c,b)$ $\bot$

$P(c,c)$ $\bot$

$\forall x Q(x)$ $\top$ @a b

$Q(a)$ $\top$

X

$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$

$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$

$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$

$P(a,b) \to Q(a)$ $\bot$

$P(a,b)$ $\top$

$Q(a)$ $\bot$

$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$

$\exists y P(c,y)$ $\bot$ @a @b @c

$P(c,a)$ $\bot$

$P(c,b)$ $\bot$

$P(c,c)$ $\bot$

$\forall x Q(x)$ $\top$ @a b

$Q(a)$ $\top$

X

$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$

$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$

$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$

$P(a,b) \to Q(a)$ $\bot$

$P(a,b)$ $\top$

$Q(a)$ $\bot$

$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$

$\exists y P(c,y)$ $\bot$ @a @b @c

$P(c,a)$ $\bot$

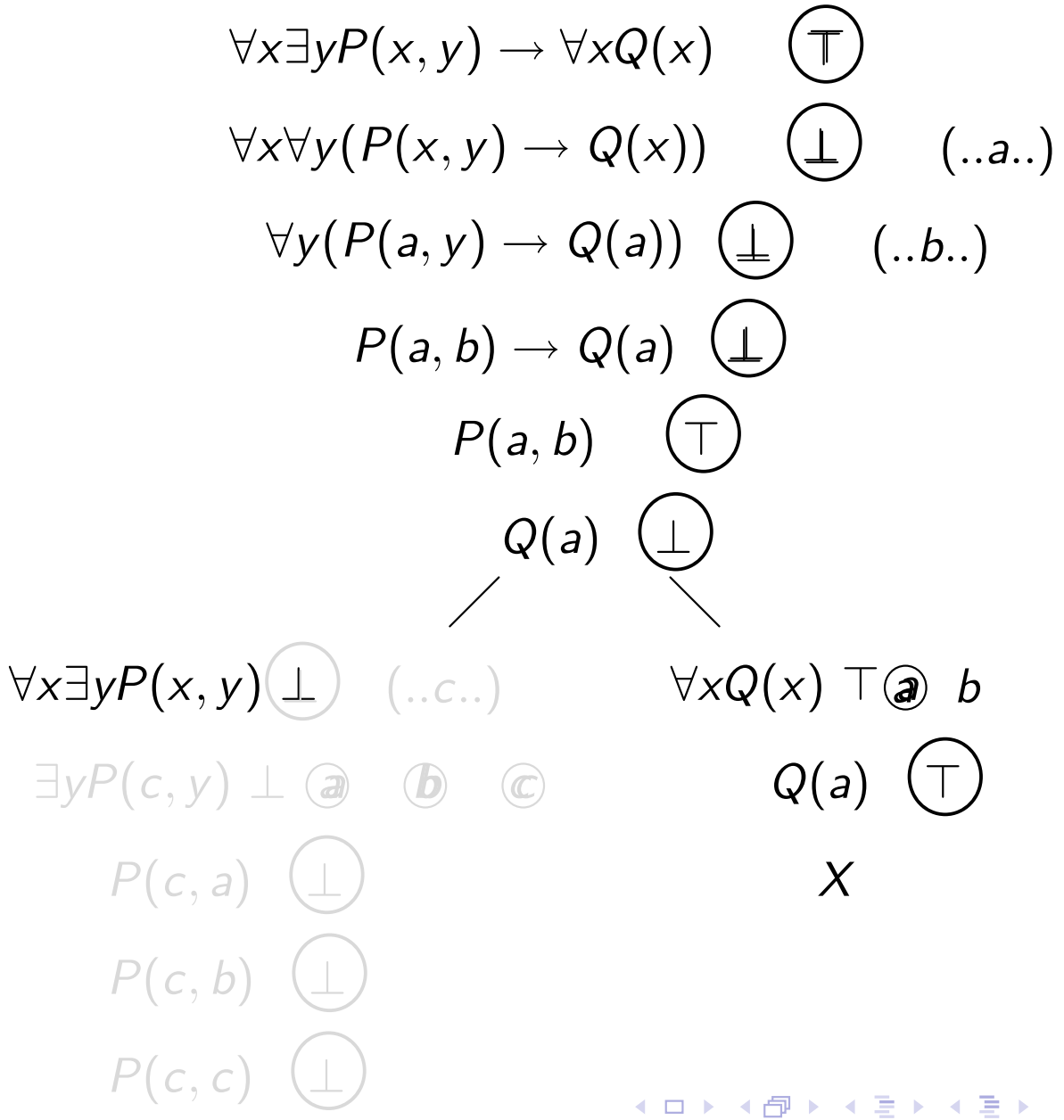
$P(c,b)$ $\bot$

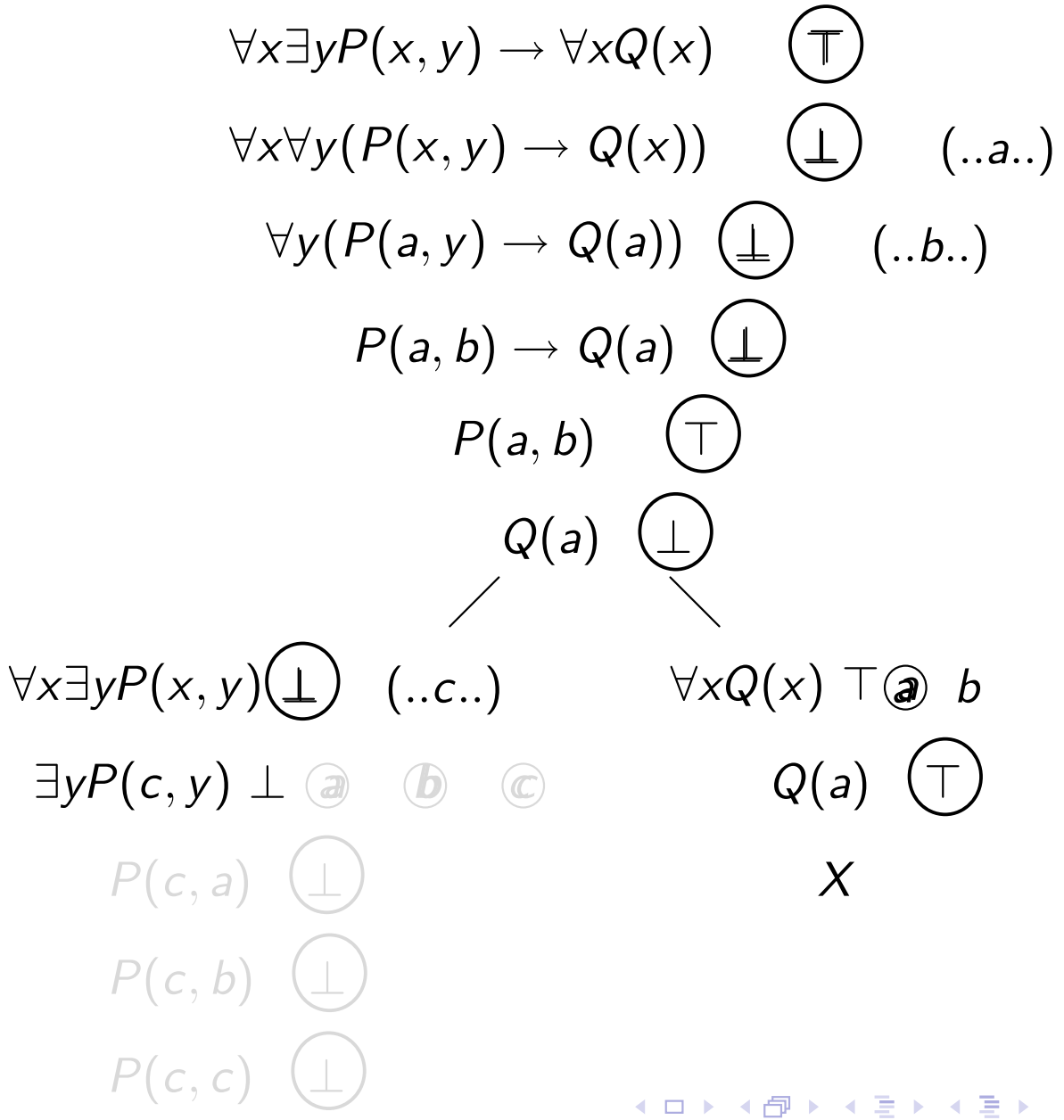
$P(c,c)$ $\bot$

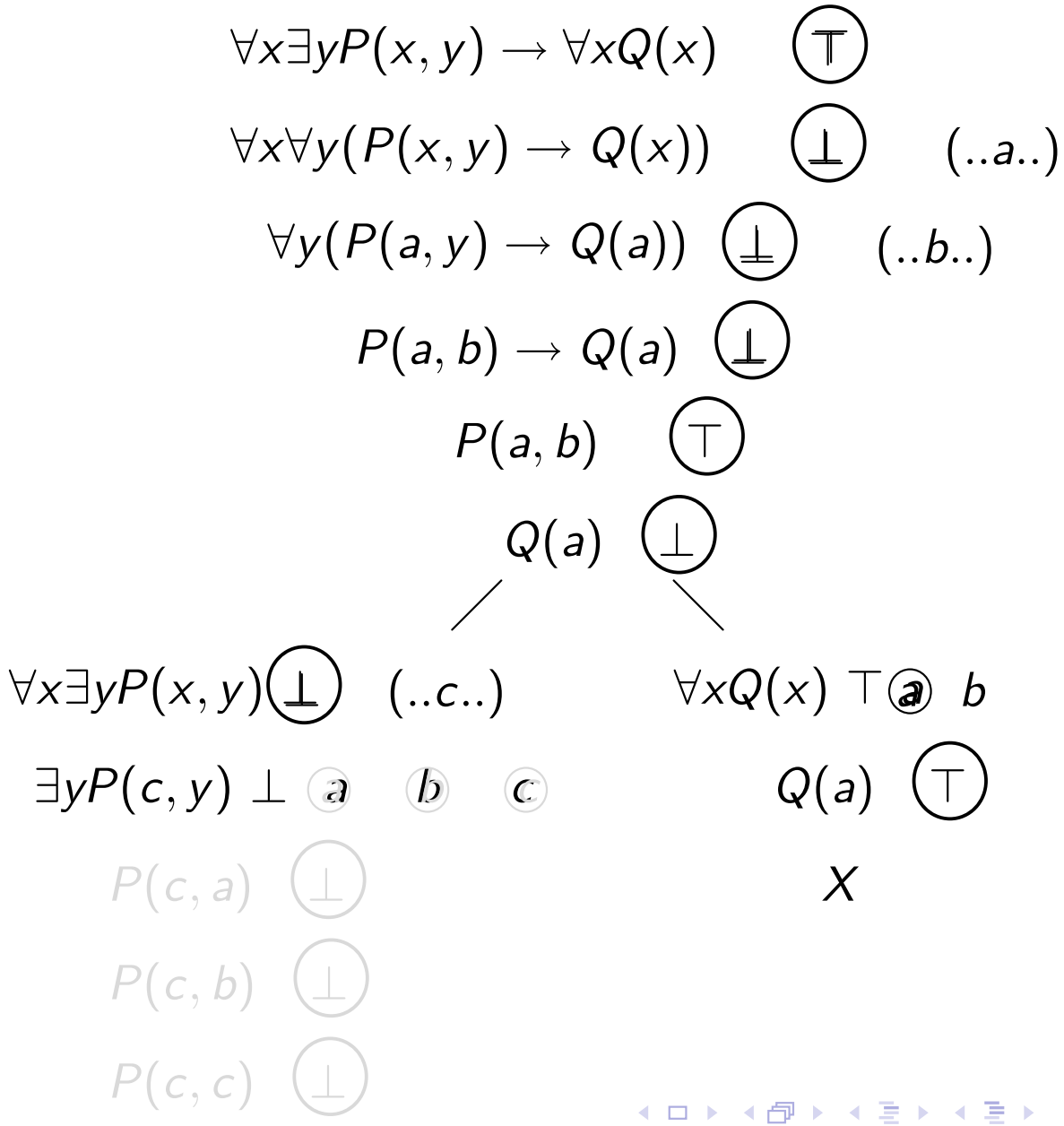
$\forall x Q(x)$ $\top$ @a b

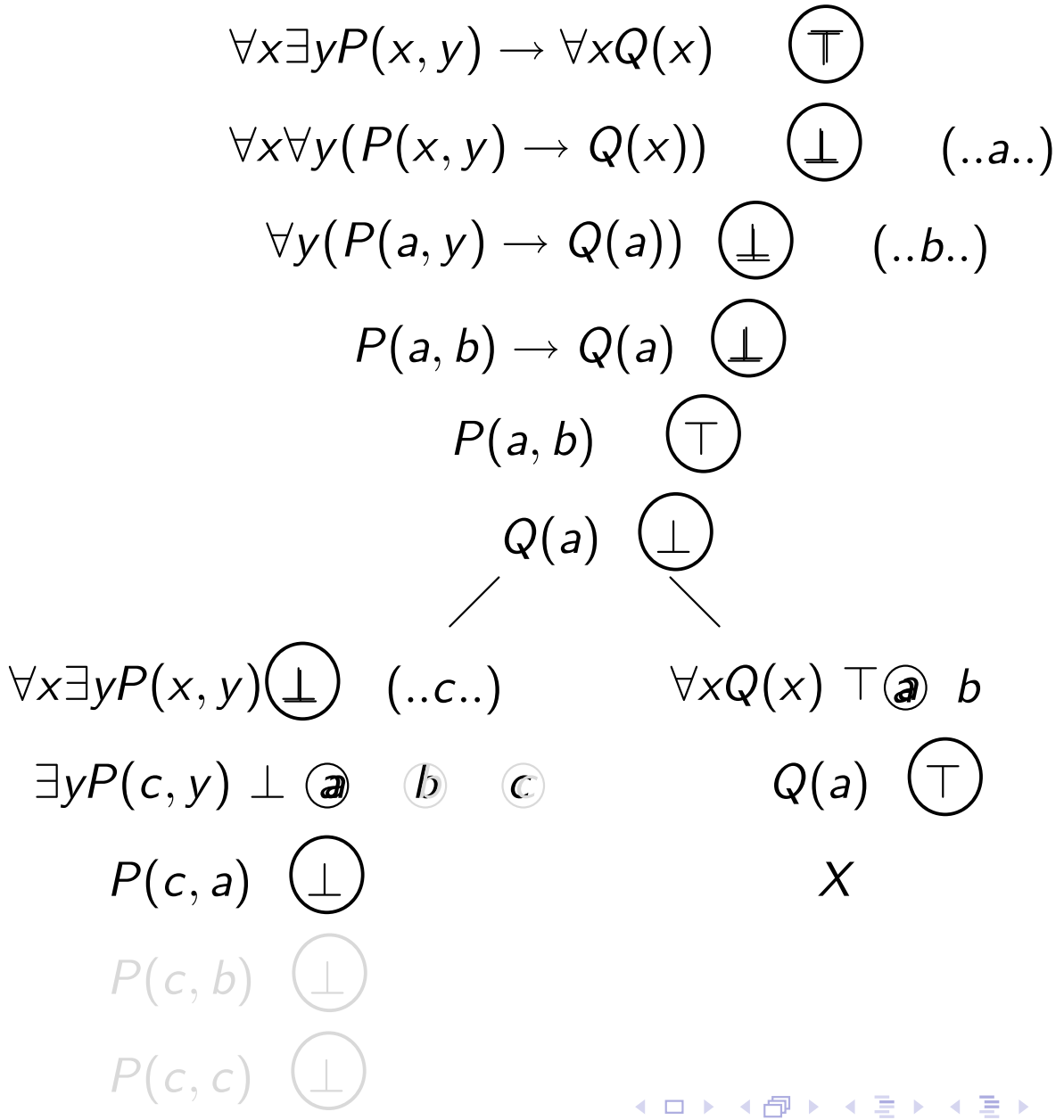
$Q(a)$ $\top$

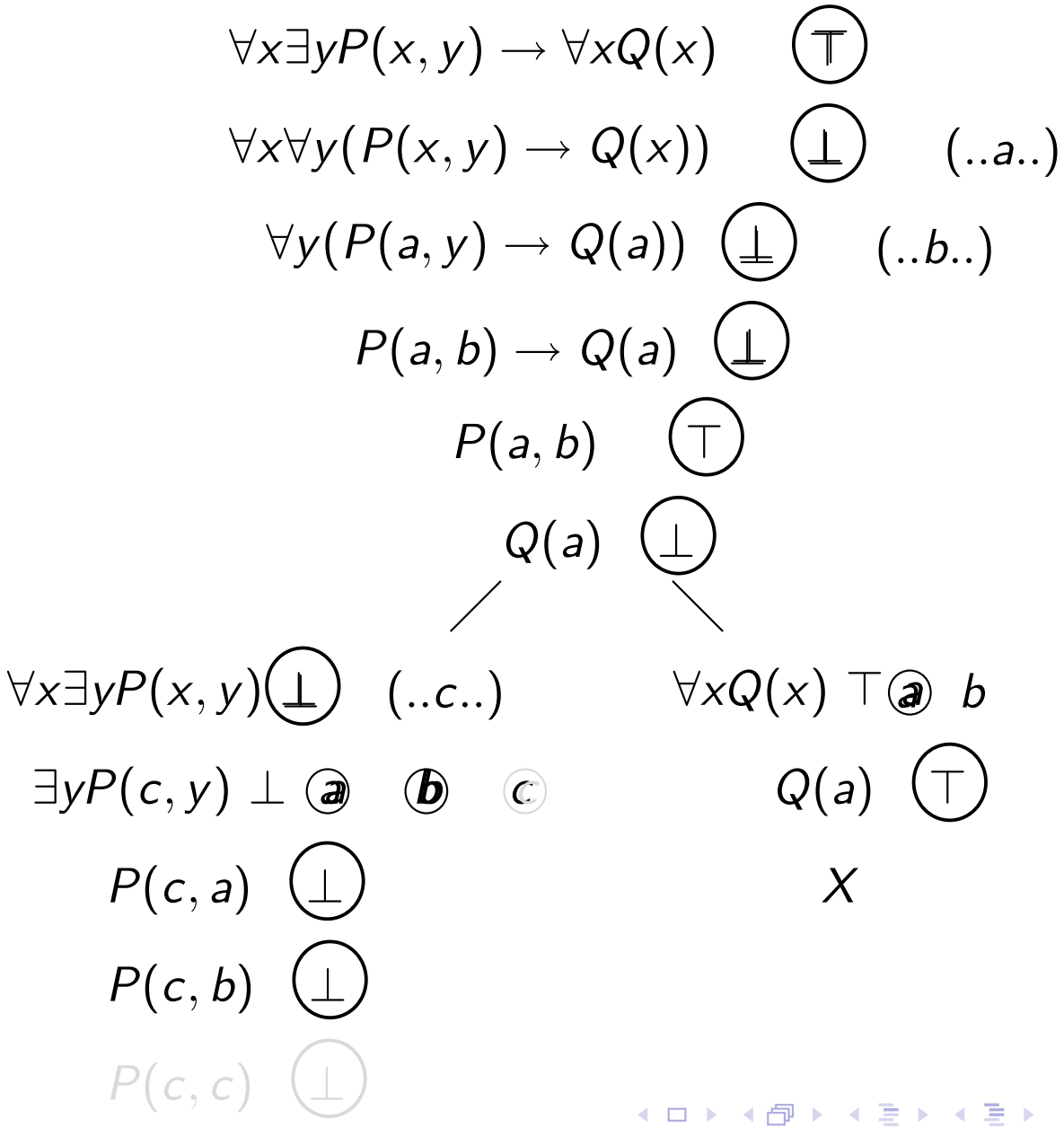
X

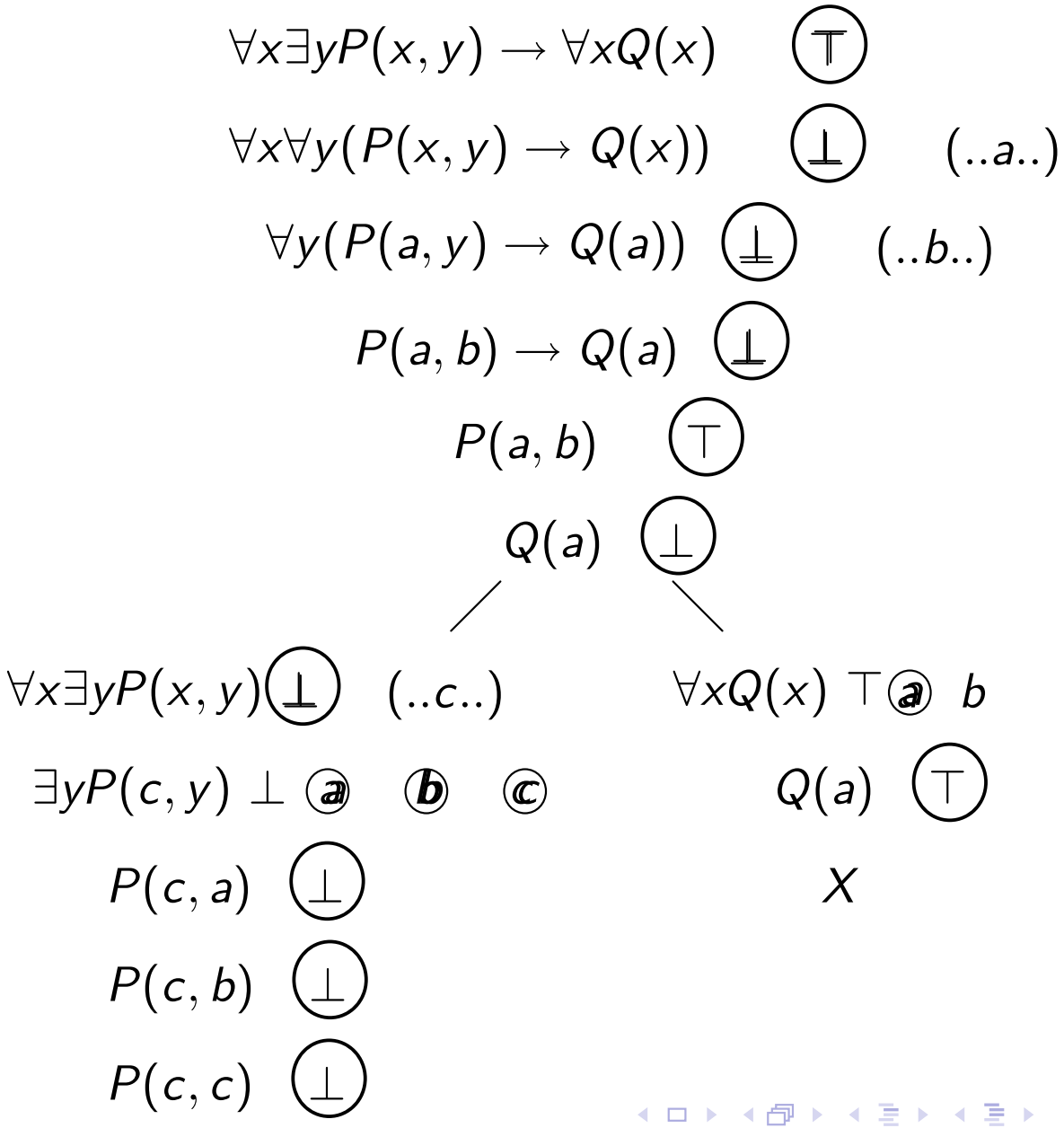

$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$
$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$
$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$
$P(a,b) \to Q(a)$ $\bot$
$P(a,b)$ $\top$
$Q(a)$ $\bot$
$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$
$\exists y P(c,y)$ $\bot$ @a @b @c
$P(c,a)$ $\bot$
$P(c,b)$ $\bot$
$P(c,c)$ $\bot$
$\forall x Q(x)$ $\top$ @a b
$Q(a)$ $\top$
X


$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$
$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$
$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$
$P(a,b) \to Q(a)$ $\bot$
$P(a,b)$ $\top$
$Q(a)$ $\bot$
$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$
$\exists y P(c,y)$ $\bot$ a b c
$P(c,a)$ $\bot$
$P(c,b)$ $\bot$
$P(c,c)$ $\bot$
$\forall x Q(x)$ $\top$ a b
$Q(a)$ $\top$
X


$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$
$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$
$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$
$P(a,b) \to Q(a)$ $\bot$
$P(a,b)$ $\top$
$Q(a)$ $\bot$
$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$
$\exists y P(c,y)$ $\bot$ a b c
$P(c,a)$ $\bot$
$P(c,b)$ $\bot$
$P(c,c)$ $\bot$
$\forall x Q(x)$ $\top$ a b
$Q(a)$ $\top$
X


$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$
$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$
$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$
$P(a,b) \to Q(a)$ $\bot$
$P(a,b)$ $\top$
$Q(a)$ $\bot$
$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$
$\exists y P(c,y)$ $\bot$ @ a b c
$P(c,a)$ $\bot$
$P(c,b)$ $\bot$
$P(c,c)$ $\bot$
$\forall x Q(x)$ $\top$ @ a b
$Q(a)$ $\top$
X


$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\mathbb{T}$
$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$
$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$
$P(a,b) \to Q(a)$ $\bot$
$P(a,b)$ $\top$
$Q(a)$ $\bot$
$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$
$\exists y P(c,y)$ $\bot$ @ a @ b c
$P(c,a)$ $\bot$
$P(c,b)$ $\bot$
$P(c,c)$ $\bot$
$\forall x Q(x)$ $\top$ @ a b
$Q(a)$ $\top$
X


$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\mathbb{T}$
$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$
$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$
$P(a,b) \to Q(a)$ $\bot$
$P(a,b)$ $\top$
$Q(a)$ $\bot$
$\forall x \exists y P(x,y)$ $\bot$ $(..c..)$
$\exists y P(c,y)$ $\bot$ @a @b @c
$P(c,a)$ $\bot$
$P(c,b)$ $\bot$
$P(c,c)$ $\bot$
$\forall x Q(x)$ $\top$ @a b
$Q(a)$ $\top$
X

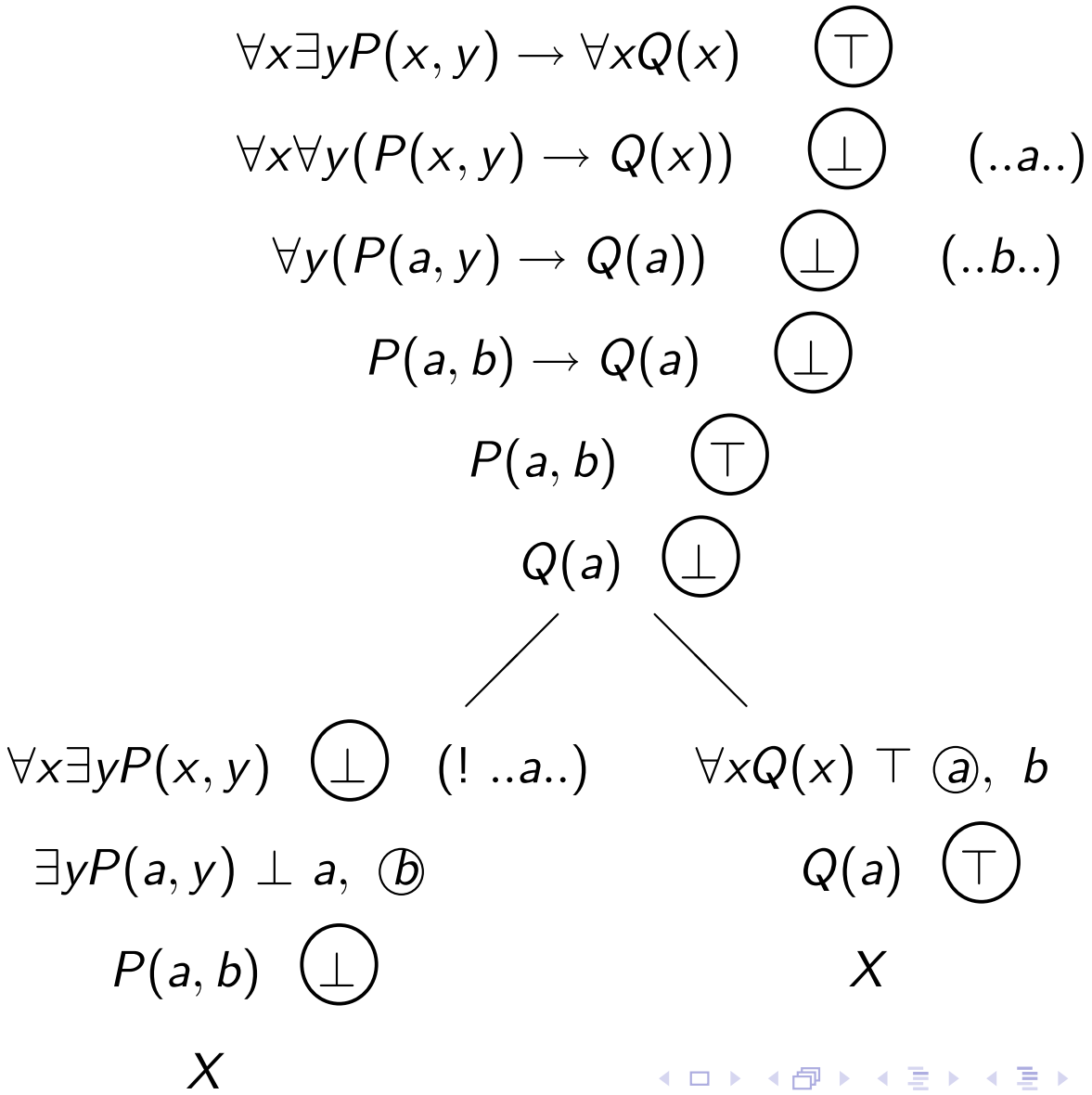
Pošto lijeva grana nije završila oznakom za kontradikciju zaključujemo da dana tvrdnja nije istinita.

S te lijeve grane možemo pročitati strukturu za koju početna tvrdnja nije istinita.

Nosač strukture je $|\mathfrak{M}| = \{a, b, c\}$, te je $P^{\mathfrak{M}} = \{(a, b)\}$ i $Q^{\mathfrak{M}} = \emptyset$.

Uočimo još da na desnoj grani prvi redak oblika $\forall x Q(x)\ \top$ nismo analizirali u odnosu na element b. To nije nužno jer smo već naišli na kontradikciju.

Pogledajmo sada što se događa kada u rješavanju gornjeg zadatka koristimo "stare" elemente.


$\forall x \exists y P(x,y) \to \forall x Q(x)$ $\top$
$\forall x \forall y (P(x,y) \to Q(x))$ $\bot$ $(..a..)$
$\forall y (P(a,y) \to Q(a))$ $\bot$ $(..b..)$
$P(a,b) \to Q(a)$ $\bot$
$P(a,b)$ $\top$
$Q(a)$ $\bot$
$\forall x \exists y P(x,y)$ $\bot$ $(!\ ..a..)$
$\exists y P(a,y)$ $\bot$ $a,\ ⓑ$
$P(a,b)$ $\bot$
X
$\forall x Q(x)$ $\top$ $ⓐ,\ b$
$Q(a)$ $\top$
X

U sedmom retku smo sa (! ..*a*..) označili da ne uvodimo novi element već koristimo stari.

Dani test je na svim granama završio kontradikcijom, pa bi brzopleto (i krivo) mogli zaključiti da je dana formula valjana.

Iz prethodnog testa znamo da formula nije valjana.

Korištenjem "starog" elementa a mi smo posljednjim testom zapravo dokazali da ne postoji strukura s točno dva elementa koja nije model za F.

U sljedećem primjeru želimo istaknuti kako se glavni test koristi za ispitivanje je li neka formula F **oboriva**.

Početni redak u testu je oblika $F\bot$. To znači da pokušavamo odrediti strukturu koja nije model za formulu F.

Primjer 3

Ispitajmo pomoću glavnog testa je li sljedeća formula oboriva:

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \rightarrow \forall x R(x,y))) \rightarrow \exists x(Q(x) \vee R(x,x)).$$

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top$$

$$(*) \quad \exists x(Q(x) \vee R(x,x)) \; \bot$$

$$\exists x \forall y R(x,y) \; \top \quad (..a..)$$

$$\forall y R(a,y) \; \top \; @a$$

$$R(a,a) \; \top$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @a$$

$$Q(a) \vee R(a,a) \; \bot$$

$$Q(a) \; \bot$$

$$R(a,a) \; \bot$$

$$X$$

$$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \quad (..b..) \; @b$$

$$(**) \quad \exists x P(b,x) \to \forall x R(x,b) \; \top$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @b$$

$$Q(b) \vee R(b,b) \; \bot$$

$$Q(b) \; \bot$$

$$R(b,b) \; \bot$$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \quad \top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \quad \top \quad (..a..)$$

$$\forall y R(a,y) \quad \top \quad @$$

$$R(a,a) \quad \top$$

$$(*) \ \exists x(Q(x) \vee R(x,x)) \quad \bot \quad @$$

$$Q(a) \vee R(a,a) \quad \bot$$

$$Q(a) \quad \bot$$

$$R(a,a) \quad \bot$$

$$X$$

$$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \quad \top \quad (..b..) \ ⓑ$$

$$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \quad \top$$

$$(*) \ \exists x(Q(x) \vee R(x,x)) \quad \bot \quad ⓑ$$

$$Q(b) \vee R(b,b) \quad \bot$$

$$Q(b) \quad \bot$$

$$R(b,b) \quad \bot$$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \quad \top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \quad \top \quad (..a..) \qquad \forall y(\exists x P(y,x) \to \forall x R(x,y)) \quad \top \quad (..b..)\ b$$

$$\forall y R(a,y) \quad \top \ a$$

$$R(a,a) \quad \top$$

$$(*) \ \exists x(Q(x) \vee R(x,x)) \quad \bot \ a$$

$$Q(a) \vee R(a,a) \quad \bot$$

$$Q(a) \quad \bot$$

$$R(a,a) \quad \bot$$

$$X$$

$$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \quad \top$$

$$(*) \ \exists x(Q(x) \vee R(x,x)) \quad \bot \ b$$

$$Q(b) \vee R(b,b) \quad \bot$$

$$Q(b) \quad \bot$$

$$R(b,b) \quad \bot$$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \textcircled{\bot}$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \textcircled{\top}$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$\exists x \forall y R(x,y) \; \textcircled{\top} \quad (..a..)$

$\forall y R(a,y) \; \top \; @a$

$R(a,a) \; \textcircled{\top}$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @a$

$Q(a) \vee R(a,a) \; \textcircled{\bot}$

$Q(a) \; \textcircled{\bot}$

$R(a,a) \; \textcircled{\bot}$

X

$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; @b$

$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @b$

$Q(b) \vee R(b,b) \; \textcircled{\bot}$

$Q(b) \; \textcircled{\bot}$

$R(b,b) \; \textcircled{\bot}$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \bigcirc\!\!\!\!\bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \;\; \bigcirc\!\!\!\!\top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$\exists x \forall y R(x,y) \; \bigcirc\!\!\!\!\top \quad (..a..)$

$\forall y R(a,y) \; \top \; @$

$R(a,a) \; \bigcirc\!\!\!\!\top$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @$

$Q(a) \vee R(a,a) \; \bigcirc\!\!\!\!\bot$

$Q(a) \; \bigcirc\!\!\!\!\bot$

$R(a,a) \; \bigcirc\!\!\!\!\bot$

X

$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; ⓑ$

$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; ⓑ$

$Q(b) \vee R(b,b) \; \bigcirc\!\!\!\!\bot$

$Q(b) \; \bigcirc\!\!\!\!\bot$

$R(b,b) \; \bigcirc\!\!\!\!\bot$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \bigcirc\!\!\!\!\bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \bigcirc\!\!\!\!\top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$\exists x \forall y R(x,y) \; \bigcirc\!\!\!\!\top \quad (..a..)$

$\forall y R(a,y) \; \top \; @$

$R(a,a) \; \bigcirc\!\!\!\!\top$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @$

$Q(a) \vee R(a,a) \; \bigcirc\!\!\!\!\bot$

$Q(a) \; \bigcirc\!\!\!\!\bot$

$R(a,a) \; \bigcirc\!\!\!\!\bot$

X

$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; ⓑ$

$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; ⓑ$

$Q(b) \vee R(b,b) \; \bigcirc\!\!\!\!\bot$

$Q(b) \; \bigcirc\!\!\!\!\bot$

$R(b,b) \; \bigcirc\!\!\!\!\bot$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \bigcirc\!\!\!\!\perp$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \bigcirc\!\!\!\!\top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \perp$$

$$\exists x \forall y R(x,y) \; \bigcirc\!\!\!\!\top \quad (..a..)$$

$$\forall y R(a,y) \; \top \; @$$

$$R(a,a) \; \bigcirc\!\!\!\!\top$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \perp \; @$$

$$Q(a) \vee R(a,a) \; \bigcirc\!\!\!\!\perp$$

$$Q(a) \; \bigcirc\!\!\!\!\perp$$

$$R(a,a) \; \bigcirc\!\!\!\!\perp$$

$$X$$

$$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; ⓑ$$

$$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \perp \; ⓑ$$

$$Q(b) \vee R(b,b) \; \bigcirc\!\!\!\!\perp$$

$$Q(b) \; \bigcirc\!\!\!\!\perp$$

$$R(b,b) \; \bigcirc\!\!\!\!\perp$$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \bigcirc\!\!\!\!\bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \bigcirc\!\!\!\!\top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$\exists x \forall y R(x,y) \; \bigcirc\!\!\!\!\top \quad (..a..)$

$\forall y R(a,y) \; \top \; @$

$R(a,a) \; \bigcirc\!\!\!\!\top$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @$

$Q(a) \vee R(a,a) \; \bigcirc\!\!\!\!\bot$

$Q(a) \; \bigcirc\!\!\!\!\bot$

$R(a,a) \; \bigcirc\!\!\!\!\bot$

X

$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; ⓑ$

$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$

$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; ⓑ$

$Q(b) \vee R(b,b) \; \bigcirc\!\!\!\!\bot$

$Q(b) \; \bigcirc\!\!\!\!\bot$

$R(b,b) \; \bigcirc\!\!\!\!\bot$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \bigcirc\!\!\!\!\bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \bigcirc\!\!\!\!\top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \; \bigcirc\!\!\!\!\top \quad (..a..) \qquad \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; ⓑ$$

$$\forall y R(a,y) \; \top \; ⓐ \qquad (**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$$

$$R(a,a) \; \bigcirc\!\!\!\!\top \qquad (*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; ⓑ$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; ⓐ \qquad Q(b) \vee R(b,b) \; \bigcirc\!\!\!\!\bot$$

$$Q(a) \vee R(a,a) \; \bigcirc\!\!\!\!\bot \qquad Q(b) \; \bigcirc\!\!\!\!\bot$$

$$Q(a) \; \bigcirc\!\!\!\!\bot \qquad R(b,b) \; \bigcirc\!\!\!\!\bot$$

$$R(a,a) \; \bigcirc\!\!\!\!\bot$$

$$X$$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \bot$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \; \top \quad (..a..) \qquad \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; ⓑ$$

$$\forall y R(a,y) \; \top \; ⓐ \qquad (**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$$

$$R(a,a) \; \top \qquad (*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; ⓑ$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; ⓐ \qquad Q(b) \vee R(b,b) \; \bot$$

$$Q(a) \vee R(a,a) \; \bot \qquad Q(b) \; \bot$$

$$Q(a) \; \bot \qquad R(b,b) \; \bot$$

$$R(a,a) \; \bot$$

$$X$$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \textcircled{\bot}$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \textcircled{\top}$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \; \textcircled{\top} \quad (..a..) \qquad \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; \textcircled{b}$$

$$\forall y R(a,y) \; \top \; \textcircled{a} \qquad (**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$$

$$R(a,a) \; \textcircled{\top} \qquad (*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; \textcircled{b}$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; \textcircled{a} \qquad Q(b) \vee R(b,b) \; \textcircled{\bot}$$

$$Q(a) \vee R(a,a) \; \textcircled{\bot} \qquad Q(b) \; \textcircled{\bot}$$

$$Q(a) \; \textcircled{\bot} \qquad R(b,b) \; \textcircled{\bot}$$

$$R(a,a) \; \textcircled{\bot}$$

$$X$$

Nastavak na sljedećem slajdu

$$(\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y))) \to \exists x(Q(x) \vee R(x,x)) \; \textcircled{\bot}$$

$$\exists x \forall y R(x,y) \vee \forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \textcircled{\top}$$

$$(*) \qquad \exists x(Q(x) \vee R(x,x)) \quad \bot$$

$$\exists x \forall y R(x,y) \; \textcircled{\top} \quad (..a..)$$

$$\forall y R(a,y) \; \top \; @a$$

$$R(a,a) \; \textcircled{\top}$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @a$$

$$Q(a) \vee R(a,a) \; \textcircled{\bot}$$

$$Q(a) \; \textcircled{\bot}$$

$$R(a,a) \; \textcircled{\bot}$$

$$X$$

$$\forall y(\exists x P(y,x) \to \forall x R(x,y)) \; \top \qquad (..b..) \; @b$$

$$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$$

$$(*) \; \exists x(Q(x) \vee R(x,x)) \; \bot \; @b$$

$$Q(b) \vee R(b,b) \; \textcircled{\bot}$$

$$Q(b) \; \textcircled{\bot}$$

$$R(b,b) \; \textcircled{\bot}$$

Nastavak na sljedećem slajdu

$$(**) \qquad \exists x P(b,x) \to \forall x R(x,b) \; \top$$

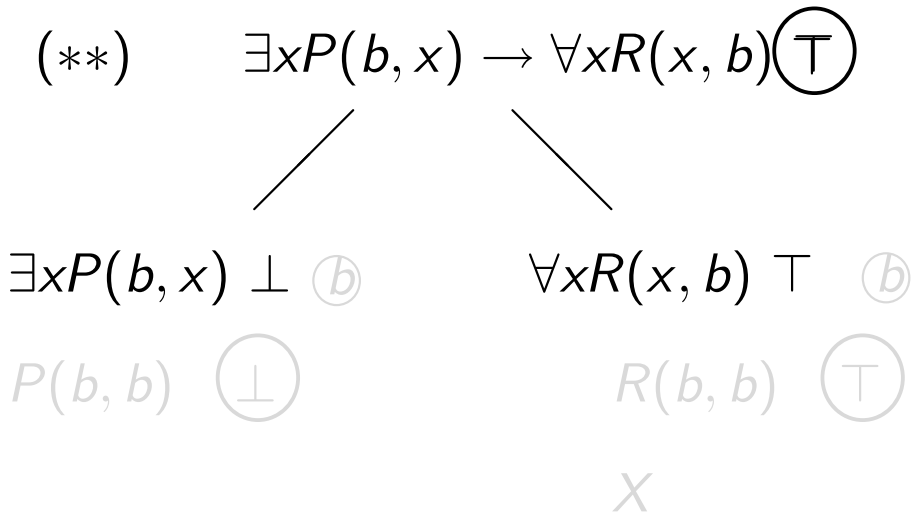
$$\exists x P(b,x) \; \bot \; (b)$$

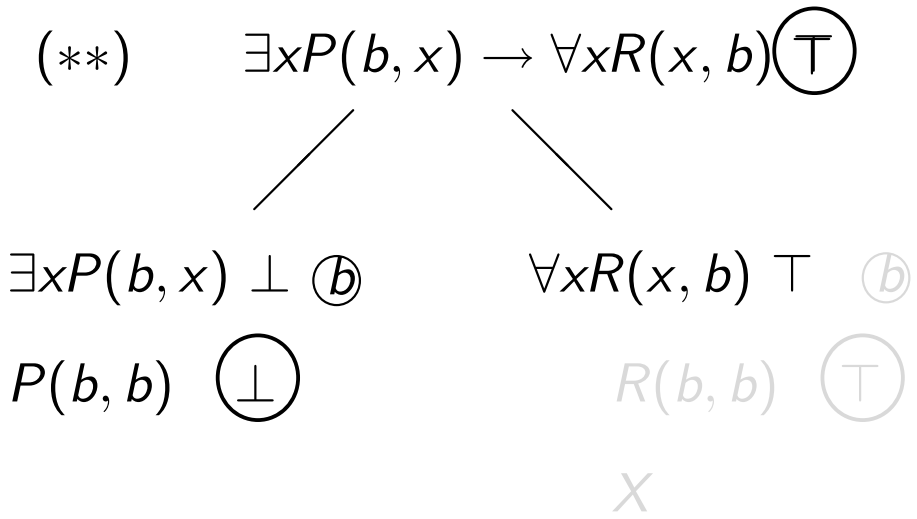
$$P(b,b) \; (\bot)$$

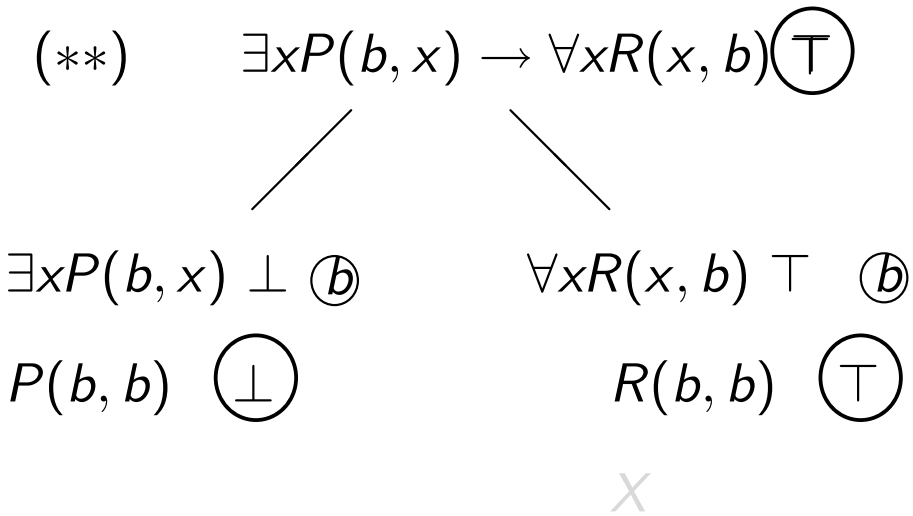
$$\forall x R(x,b) \; \top \; (b)$$

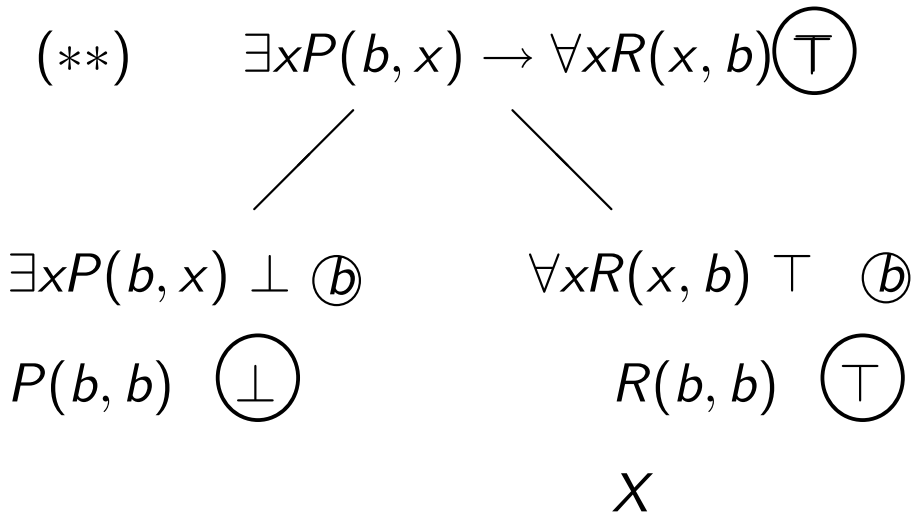
$$R(b,b) \; (\top)$$

$$X$$


$(**)$
$\exists x P(b,x) \to \forall x R(x,b)$ $\top$
$\exists x P(b,x)$ $\bot$ b
$P(b,b)$ $\bot$
$\forall x R(x,b)$ $\top$ b
$R(b,b)$ $\top$
X


$(**)$
$\exists x P(b,x) \to \forall x R(x,b)$ $\top$
$\exists x P(b,x)$ $\bot$ b
$P(b,b)$ $\bot$
$\forall x R(x,b)$ $\top$ b
$R(b,b)$ $\top$
X


(**)
$\exists x P(b,x) \to \forall x R(x,b)$ ⊤
$\exists x P(b,x)$ ⊥ b
$P(b,b)$ ⊥
$\forall x R(x,b)$ ⊤ b
$R(b,b)$ ⊤
X


(**)
$\exists x P(b,x) \to \forall x R(x,b)$ ⊤
$\exists x P(b,x)$ ⊥ b
$P(b,b)$ ⊥
$\forall x R(x,b)$ ⊤ b
$R(b,b)$ ⊤
X

Zadana formula je oboriva.

JEdna struktura $\mathfrak{M}$ koja to dokazuje zadana je sa: $|\mathfrak{M}| = \{b\}$, te $Q^{\mathfrak{M}} = R^{\mathfrak{M}} = P^{\mathfrak{M}} = \emptyset$.

Sljedeći primjer pokazuje da nekad test ne mora završiti, ali mi ipak možemo odrediti jednu traženu (beskonačnu) strukturu.

Primjer 4

Ispitajmo je li formula $\forall x \exists y A(x, y) \to \exists y \forall x A(x, y)$ *valjana.*

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)$ $\bot$

$\forall x \exists y A(x,y)$ $\top$ $(..a_1..)$ a_1 a_2 a_3 a_4 a_5

$\exists y \forall x A(x,y)$ $\bot$ a_1 a_2 a_3 a_4 a_5

$\exists y A(a_1,y)$ $\top$ $(..a_2..)$

$\forall x A(x,a_1)$ $\bot$ $(..a_3..)$

$A(a_1,a_2)$ $\top$

$A(a_3,a_1)$ $\bot$

$\exists y A(a_2,y)$ $\top$ $(..a_4..)$

$\forall x A(x,a_2)$ $\bot$ $(..a_5..)$

$A(a_2,a_4)$ $\top$

$A(a_5,a_2)$ $\bot$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)\ (\bot)$

$\forall x \exists y A(x,y)\ \top \quad (..a_1..)\ a_1\ a_2\ a_3\ a_4\ a_5$

$\exists y \forall x A(x,y)\ \bot \quad a_1\ a_2\ a_3\ a_4\ a_5$

$\exists y A(a_1,y)\ (\top) \quad (..a_2..)$

$\forall x A(x,a_1)\ (\bot) \quad (..a_3..)$

$A(a_1,a_2)\ (\top)$

$A(a_3,a_1)\ (\bot)$

$\exists y A(a_2,y)\ (\top) \quad (..a_4..)$

$\forall x A(x,a_2)\ (\bot) \quad (..a_5..)$

$A(a_2,a_4)\ (\top)$

$A(a_5,a_2)\ (\bot)$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)\ (\bot)$

$\forall x \exists y A(x,y)\ \top \quad (..a_1..)\ (a_1)\ a_2\ a_3\ a_4\ a_5$

$\exists y \forall x A(x,y)\ \bot \quad a_1\ a_2\ a_3\ a_4\ a_5$

$\exists y A(a_1,y)\ (\top) \quad (..a_2..)$

$\forall x A(x,a_1)\ (\bot) \quad (..a_3..)$

$A(a_1,a_2)\ (\top)$

$A(a_3,a_1)\ (\bot)$

$\exists y A(a_2,y)\ (\top) \quad (..a_4..)$

$\forall x A(x,a_2)\ (\bot) \quad (..a_5..)$

$A(a_2,a_4)\ (\top)$

$A(a_5,a_2)\ (\bot)$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)\ (\bot)$

$\forall x \exists y A(x,y)\ \top \quad (..a_1..)\ (a_1)\ a_2\ a_3\ a_4\ a_5$

$\exists y \forall x A(x,y)\ \bot \quad (a_1)\ a_2\ a_3\ a_4\ a_5$

$\exists y A(a_1,y)\ (\top) \quad (..a_2..)$

$\forall x A(x,a_1)\ (\bot) \quad (..a_3..)$

$A(a_1,a_2)\ (\top)$

$A(a_3,a_1)\ (\bot)$

$\exists y A(a_2,y)\ (\top) \quad (..a_4..)$

$\forall x A(x,a_2)\ (\bot) \quad (..a_5..)$

$A(a_2,a_4)\ (\top)$

$A(a_5,a_2)\ (\bot)$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)\ (\bot)$

$\forall x \exists y A(x,y)\ \top \quad (..a_1..)\ (a_1)\ a_2\ a_3\ a_4\ a_5$

$\exists y \forall x A(x,y)\ \bot \quad (a_1)\ a_2\ a_3\ a_4\ a_5$

$\exists y A(a_1,y)\ (\top) \quad (..a_2..)$

$\forall x A(x,a_1)\ (\bot) \quad (..a_3..)$

$A(a_1,a_2)\ (\top)$

$A(a_3,a_1)\ (\bot)$

$\exists y A(a_2,y)\ (\top) \quad (..a_4..)$

$\forall x A(x,a_2)\ (\bot) \quad (..a_5..)$

$A(a_2,a_4)\ (\top)$

$A(a_5,a_2)\ (\bot)$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)\ (\bot)$

$\forall x \exists y A(x,y)\ \top \quad (..a_1..)\ (a_1)\ a_2\ a_3\ a_4\ a_5$

$\exists y \forall x A(x,y)\ \bot \quad (a_1)\ a_2\ a_3\ a_4\ a_5$

$\exists y A(a_1,y)\ (\top) \quad (..a_2..)$

$\forall x A(x,a_1)\ (\bot\!\!\bot) \quad (..a_3..)$

$A(a_1,a_2)\ (\top)$

$A(a_3,a_1)\ (\bot)$

$\exists y A(a_2,y)\ (\top) \quad (..a_4..)$

$\forall x A(x,a_2)\ (\bot\!\!\bot) \quad (..a_5..)$

$A(a_2,a_4)\ (\top)$

$A(a_5,a_2)\ (\bot)$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)$ $\bot$

$\forall x \exists y A(x,y)$ $\top$ $(..a_1..)$ a_1 a_2 a_3 a_4 a_5

$\exists y \forall x A(x,y)$ $\bot$ a_1 a_2 a_3 a_4 a_5

$\exists y A(a_1,y)$ $\top$ $(..a_2..)$

$\forall x A(x,a_1)$ $\bot$ $(..a_3..)$

$A(a_1,a_2)$ $\top$

$A(a_3,a_1)$ $\bot$

$\exists y A(a_2,y)$ $\top$ $(..a_4..)$

$\forall x A(x,a_2)$ $\bot$ $(..a_5..)$

$A(a_2,a_4)$ $\top$

$A(a_5,a_2)$ $\bot$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)$ $\bot$

$\forall x \exists y A(x,y)$ $\top$ $(..a_1..)$ a_1 a_2 a_3 a_4 a_5

$\exists y \forall x A(x,y)$ $\bot$ a_1 a_2 a_3 a_4 a_5

$\exists y A(a_1,y)$ $\top$ $(..a_2..)$

$\forall x A(x,a_1)$ $\bot$ $(..a_3..)$

$A(a_1,a_2)$ $\top$

$A(a_3,a_1)$ $\bot$

$\exists y A(a_2,y)$ $\top$ $(..a_4..)$

$\forall x A(x,a_2)$ $\bot$ $(..a_5..)$

$A(a_2,a_4)$ $\top$

$A(a_5,a_2)$ $\bot$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)\ (\bot)$

$\forall x \exists y A(x,y)\ \top \quad (..a_1..)\ (a_1)\ (a_2)\ a_3\ a_4\ a_5$

$\exists y \forall x A(x,y)\ \bot \quad (a_1)\ (a_2)\ a_3\ a_4\ a_5$

$\exists y A(a_1,y)\ (\top) \quad (..a_2..)$

$\forall x A(x,a_1)\ (\bot) \quad (..a_3..)$

$A(a_1,a_2)\ (\top)$

$A(a_3,a_1)\ (\bot)$

$\exists y A(a_2,y)\ (\top) \quad (..a_4..)$

$\forall x A(x,a_2)\ (\bot) \quad (..a_5..)$

$A(a_2,a_4)\ (\top)$

$A(a_5,a_2)\ (\bot)$

$\vdots$

$\forall x \exists y A(x,y) \to \exists y \forall x A(x,y)$ $\bot$

$\forall x \exists y A(x,y)$ $\top$ $(..a_1..)$ a_1 a_2 a_3 a_4 a_5

$\exists y \forall x A(x,y)$ $\bot$ a_1 a_2 a_3 a_4 a_5

$\exists y A(a_1,y)$ $\top$ $(..a_2..)$

$\forall x A(x,a_1)$ $\bot$ $(..a_3..)$

$A(a_1,a_2)$ $\top$

$A(a_3,a_1)$ $\bot$

$\exists y A(a_2,y)$ $\top$ $(..a_4..)$

$\forall x A(x,a_2)$ $\bot$ $(..a_5..)$

$A(a_2,a_4)$ $\top$

$A(a_5,a_2)$ $\bot$

$\forall x \exists y A(x,y) \longrightarrow \exists y \forall x A(x,y) \; (\bot)$

$\forall x \exists y A(x,y) \; \top \quad (..a_1..) \; (a_1) \; (a_2) \; a_3 \; a_4 \; a_5$

$\exists y \forall x A(x,y) \; \bot \quad (a_1) \; (a_2) \; a_3 \; a_4 \; a_5$

$\exists y A(a_1,y) \; (\top) \quad (..a_2..)$

$\forall x A(x,a_1) \; (\bot\!\!\bot) \quad (..a_3..)$

$A(a_1,a_2) \; (\top)$

$A(a_3,a_1) \; (\bot)$

$\exists y A(a_2,y) \; (\top) \quad (..a_4..)$

$\forall x A(x,a_2) \; (\bot\!\!\bot) \quad (..a_5..)$

$A(a_2,a_4) \; (\top)$

$A(a_5,a_2) \; (\bot)$

$\vdots$

Neka je $|\mathfrak{M}| = \{a_n : n \in \mathbb{N} \setminus \{0\}\}$, te $A^{\mathfrak{M}} = \{(a_n, a_{2n}) : n \in \mathbb{N}\}$

Nije teško vidjeti da vrijedi $\mathfrak{M} \not\models \forall x \exists y A(x, y) \rightarrow \exists y \forall x A(x, y)$.

To znači da dana formula nije valjana.

Prethodni primjer je prije svega važan kako bi istaknuli da glavni test ne mora uopće završti.

Istaknimo samo da je **moguće konstruirati** konačnu strukturu koja nije model za danu formulu.

Churchov teorem

Formula

$$\forall x_1 \forall x_2 \forall x_3 (R(x_1, x_1) \wedge (R(x_1, x_3) \rightarrow (R(x_1, x_2) \vee R(x_2, x_3)))) \rightarrow$$
$$\exists y \forall z R(y, z)$$

je primjer formule za koju je svaka konačna struktura model, ali ona ipak nije valjana. (Pokušajte to dokazati pomoću glavnog testa.)

Analiza dosta brzo postaje jako složena.

Nije uopće jasna strategija kojom bismo konstruirali beskonačnu strukturu.

Pokušajte zatim pomoću glavnog testa ispitati slijedi li logički formula

$$\forall x \forall y \exists z (R(x,y) \to (R(x,z) \to R(z,y)))$$

iz skupa formula

$$\{\forall x \exists y R(x,y),\ \forall x \forall y (R(x,y) \to \neg R(y,x)\}.$$

Analiza vrlo brzo postaje jako složena, te nije jasno hoće li glavni test uopće završiti.

Naravno, vrlo lako je napisati još kompliciranije formule za koje će ispitivanje valjanosti biti jako složeno.

No, to nije slučaj samo s glavnim testom, već se isti problemi javljaju kod svakog testa za ispitivanje valjanosti formula logike prvog reda.

To ističemo u sljedećom teoremu Alonsa Churcha.

Teorem 1 (Churchov teorem)

Logika prvog reda je neodlučiva, tj. ne postoji test kojim bi se za svaku formulu u konačno mnogo koraka mogli ispitati je li valjana.

Za dokaz prethodnog teorema morali bismo prvo uvesti osnovne pojmove i rezultate teorije izračunljivosti.

O tome ćemo govoriti kasnije, tj. u trećem ciklusu ovih predvanja.

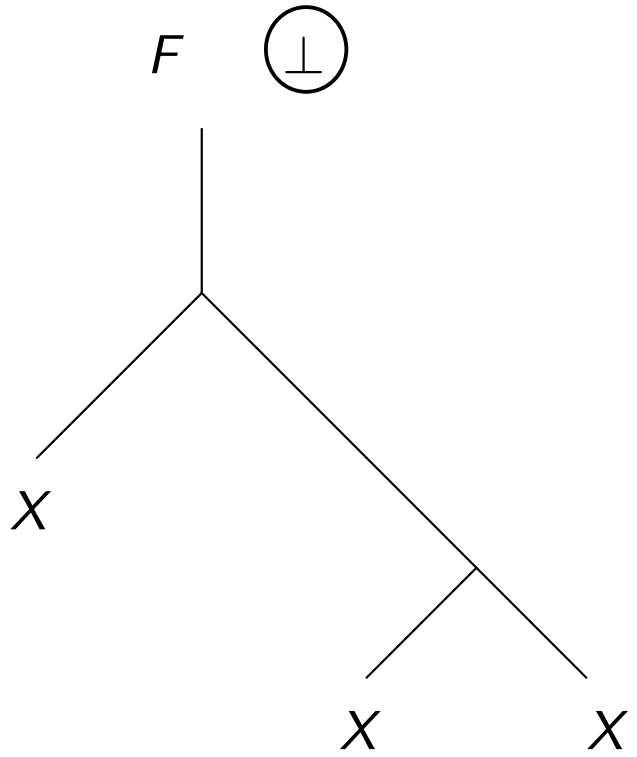
Rezimirajmo na kraju kako sve glavni test može završiti prilikom ispitivanja valjanosti neke formule.

Moguće su sljedeće dvije situacije:

a) Test je završio u konačno mnogo koraka i sve grane su završile kontradikcijom.

b) Postoji grana koja nije završila kontradikcijom.

Jednu situaciju iz slučaja a) prikazujemo sljedećom slikom.



Tada zaključujemo da je dana formula F valjana.

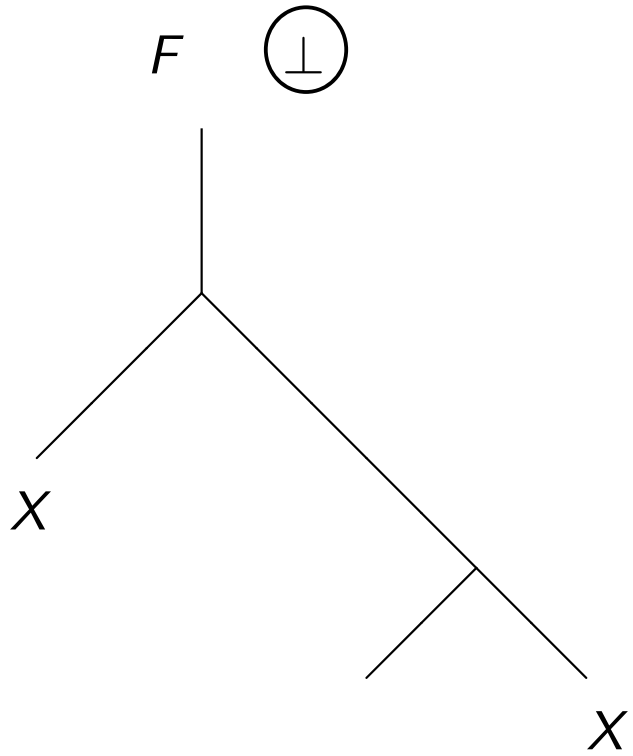
b) Postoji grana koja nije završila kontradikcijom.

Tu razlikujemo sljedeća dva podslučaja.

b_1) Postoji grana koja nije završila kontradikcijom gdje je test proveden do kraja.

b_2) Test nije proveden do kraja i nije jasno hoće li završiti u konačno koraka.

b_1) Na sljedećoj slici je prikazana jedna takva situacija.



Tada zaključujemo da je dana formula F oboriva.

S grane koja nije završila kontradikcijom čitamo strukturu koja nije model za danu formulu.

b_2) Test nije proveden do kraja i nije jasno hoće li završiti u konačno koraka.

U nekim specijalnim slučajevima moguće je na osnovu periodičkog ponavljanja odrediti beskonačnu strukturu koja nije model za danu formulu.

No, većinom u takvim slučajevima ne možemo ništa zaključiti.

Zadaci

1. Odredite preneksnu normalnu formu formule i ispitajte valjanost pomoću glavnog testa

$$(\exists z F(z) \ \wedge \ (\exists y F(y) \rightarrow \forall x G(x))) \rightarrow \exists w (F(w) \wedge G(w))$$

Rješenje. Određujemo prvo preneksnu normalnu formu dane formule:

$$(\exists z F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ \forall x \forall y (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z \forall x \forall y (F(z) \ \wedge \ (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\forall z \exists x \exists y \exists w [(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))]$$

Na posljednju formulu primjenjujemo glavni test.

Rješenje. Određujemo prvo preneksnu normalnu formu dane formule:

$$(\exists z F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ \forall x \forall y (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z \forall x \forall y (F(z) \ \wedge \ (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\forall z \exists x \exists y \exists w [(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))]$$

Na posljednju formulu primjenjujemo glavni test.

Rješenje. Određujemo prvo preneksnu normalnu formu dane formule:

$$(\exists z F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ \forall x \forall y (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z \forall x \forall y (F(z) \ \wedge \ (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\forall z \exists x \exists y \exists w [(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))]$$

Na posljednju formulu primjenjujemo glavni test.

Rješenje. Određujemo prvo preneksnu normalnu formu dane formule:

$$(\exists z F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ (\exists y F(y) \to \forall x G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ \forall x \forall y (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z \forall x \forall y (F(z) \ \wedge \ (F(y) \to G(x))) \to \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\forall z \exists x \exists y \exists w [(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))]$$

Na posljednju formulu primjenjujemo glavni test.

Rješenje. Određujemo prvo preneksnu normalnu formu dane formule:

$$(\exists z F(z) \ \wedge \ (\exists y F(y) \rightarrow \forall x G(x))) \rightarrow \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ (\exists y F(y) \rightarrow \forall x G(x))) \rightarrow \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z (F(z) \ \wedge \ \forall x \forall y (F(y) \rightarrow G(x))) \rightarrow \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\exists z \forall x \forall y (F(z) \ \wedge \ (F(y) \rightarrow G(x))) \rightarrow \exists w (F(w) \wedge G(w)) \Leftrightarrow$$

$$\forall z \exists x \exists y \exists w [(F(z) \ \wedge \ (F(y) \rightarrow G(x))) \rightarrow (F(w) \wedge G(w))]$$

Na posljednju formulu primjenjujemo glavni test.

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \bot \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \bot\,@$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot\,@$$

$$\exists w[(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot\,@$$

$$(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(a) \wedge G(a)) \quad \perp\!\!\!\perp$$

$$F(a) \ \wedge \ (F(a) \to G(a)) \quad \top\!\!\!\top$$

$$F(a) \wedge G(a) \quad \bot$$

$$F(a) \quad \top\!\!\!\top$$

$$F(a) \to G(a) \quad \top$$

Nastavak na sljedećem slajdu

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \ \mathbb{1} \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \bot \, a$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot \, a$$

$$\exists w[(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot \, a$$

$$(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(a) \wedge G(a)) \ \mathbb{1}$$

$$F(a) \ \wedge \ (F(a) \to G(a)) \ \mathbb{T}$$

$$F(a) \wedge G(a) \quad \bot$$

$$F(a) \ \mathbb{T}$$

$$F(a) \to G(a) \quad \top$$

Nastavak na sljedećem slajdu

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \mathbb{1} \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \bot\,@a$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot\,@a$$

$$\exists w[(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot\,@a$$

$$(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(a) \wedge G(a)) \quad \mathbb{1}$$

$$F(a) \ \wedge \ (F(a) \to G(a)) \quad \mathbb{T}$$

$$F(a) \wedge G(a) \quad \bot$$

$$F(a) \quad \mathbb{T}$$

$$F(a) \to G(a) \quad \top$$

Nastavak na sljedećem slajdu

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \mathbb{1} \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(a))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists w[(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(a) \wedge G(a)) \quad \mathbb{1}$$

$$F(a) \ \wedge \ (F(a) \to G(a)) \quad \mathbb{T}$$

$$F(a) \wedge G(a) \quad \perp$$

$$F(a) \quad \mathbb{T}$$

$$F(a) \to G(a) \quad \top$$

Nastavak na sljedećem slajdu

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \rightarrow G(x))) \rightarrow (F(w) \wedge G(w))] \ \textcircled{\perp} \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \rightarrow G(x))) \rightarrow (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \rightarrow G(a))) \rightarrow (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists w[(F(a) \ \wedge \ (F(a) \rightarrow G(a))) \rightarrow (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$(F(a) \ \wedge \ (F(a) \rightarrow G(a))) \rightarrow (F(a) \wedge G(a)) \ \textcircled{\perp}$$

$$F(a) \ \wedge \ (F(a) \rightarrow G(a)) \ \textcircled{\top}$$

$$F(a) \wedge G(a) \quad \perp$$

$$F(a) \ \textcircled{\top}$$

$$F(a) \rightarrow G(a) \quad \top$$

Nastavak na sljedećem slajdu

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \bot \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \bot\,@$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot\,@$$

$$\exists w[(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(w) \wedge G(w))] \quad \bot\,@$$

$$(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(a) \wedge G(a)) \quad \bot$$

$$F(a) \ \wedge \ (F(a) \to G(a)) \quad \top$$

$$F(a) \wedge G(a) \quad \bot$$

$$F(a) \quad \top$$

$$F(a) \to G(a) \quad \top$$

Nastavak na sljedećem slajdu

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \ \textcircled{\perp} \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(a))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists w[(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(a) \wedge G(a)) \ \textcircled{\perp}$$

$$F(a) \ \wedge \ (F(a) \to G(a)) \ \textcircled{\top}$$

$$F(a) \wedge G(a) \quad \perp$$

$$F(a) \ \textcircled{\top}$$

$$F(a) \to G(a) \quad \top$$

Nastavak na sljedećem slajdu

$$\forall z \exists x \exists y \exists w[(F(z) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \ \textcircled{\perp} \quad (..a..)$$

$$\exists x \exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(x))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists y \exists w[(F(a) \ \wedge \ (F(y) \to G(a))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$\exists w[(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(w) \wedge G(w))] \quad \perp \textcircled{a}$$

$$(F(a) \ \wedge \ (F(a) \to G(a))) \to (F(a) \wedge G(a)) \ \textcircled{\perp}$$

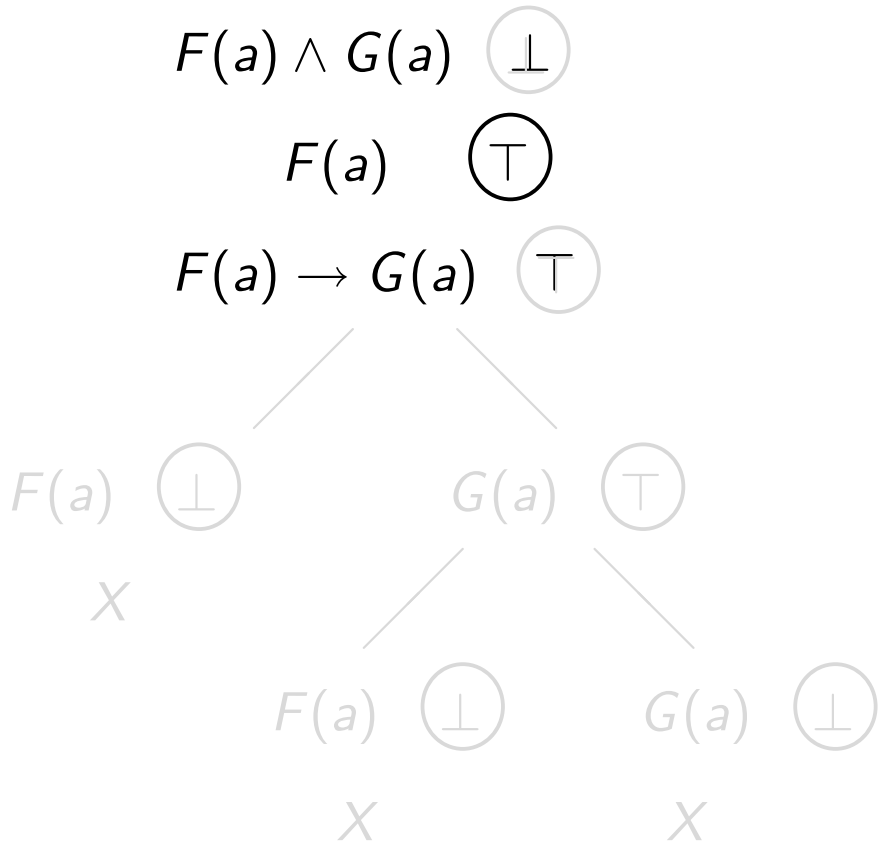
$$F(a) \ \wedge \ (F(a) \to G(a)) \ \textcircled{\top}$$

$$F(a) \wedge G(a) \quad \perp$$

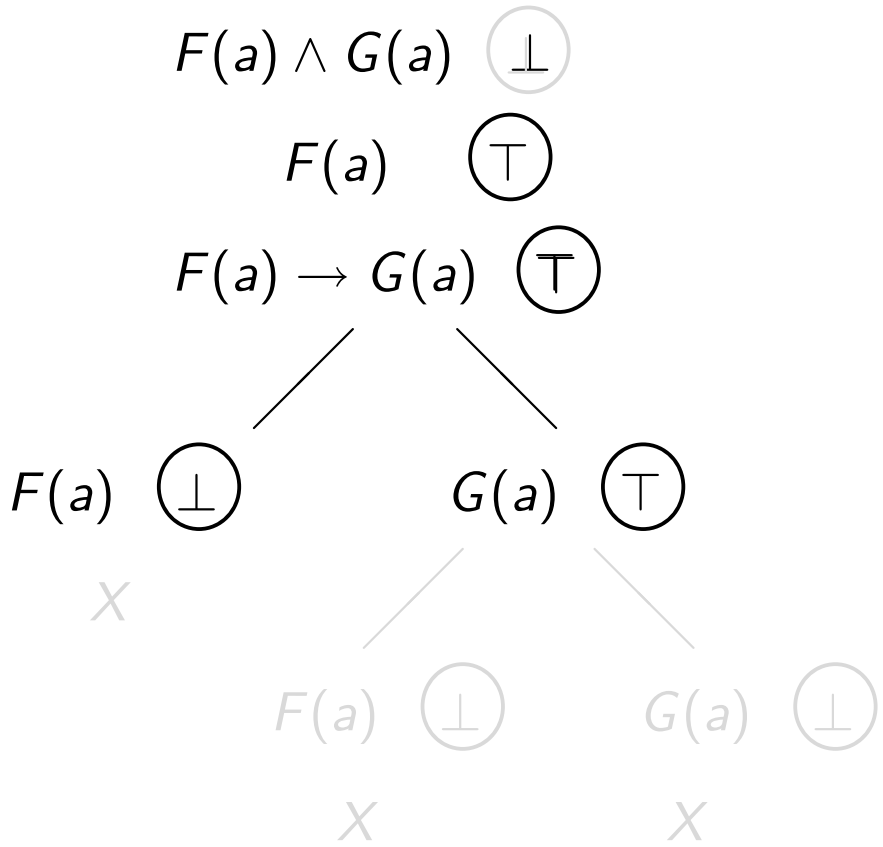
$$F(a) \ \textcircled{\top}$$

$$F(a) \to G(a) \quad \top$$

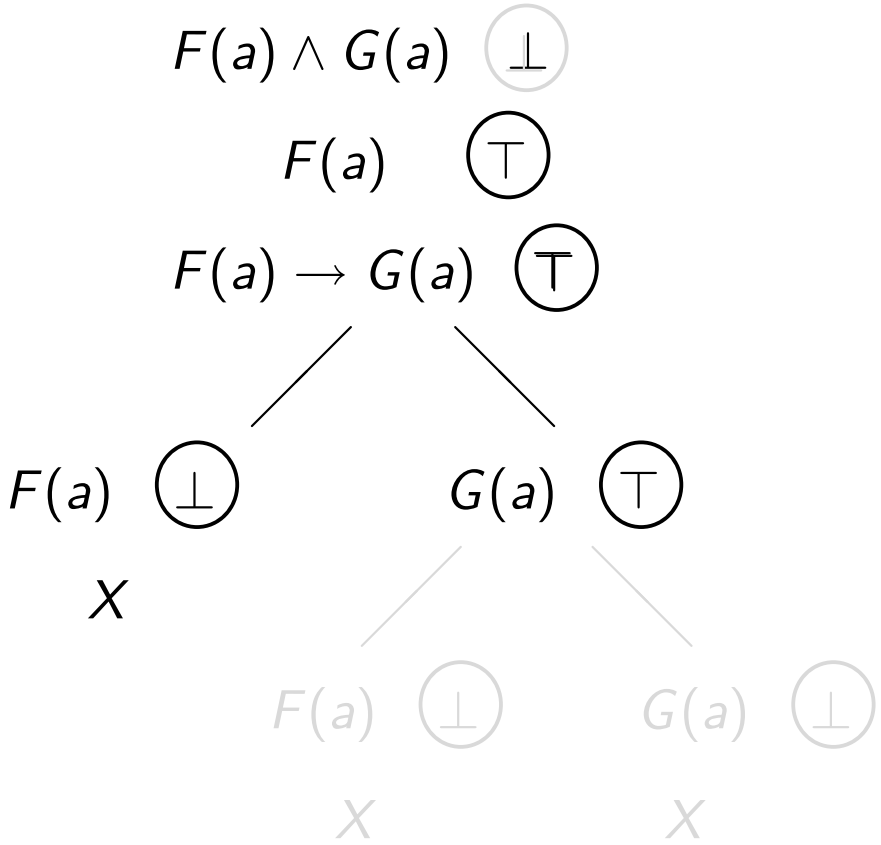
Nastavak na sljedećem slajdu



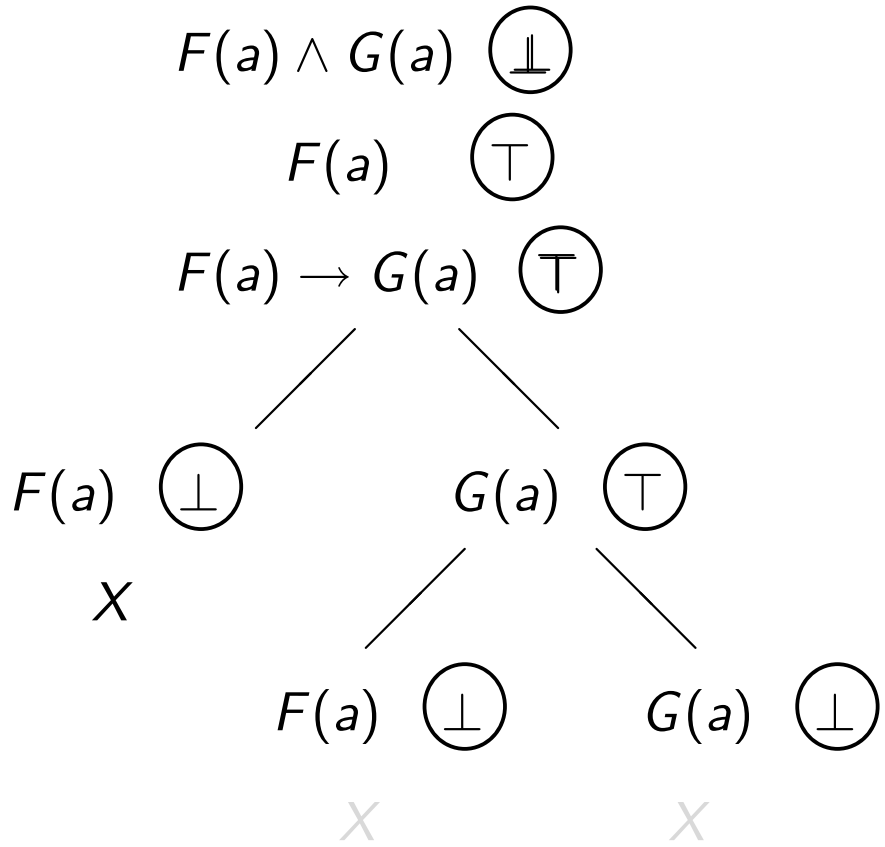
Sve grane su završile kontradikcijom pa zaključujemo da je dana formula valjana.



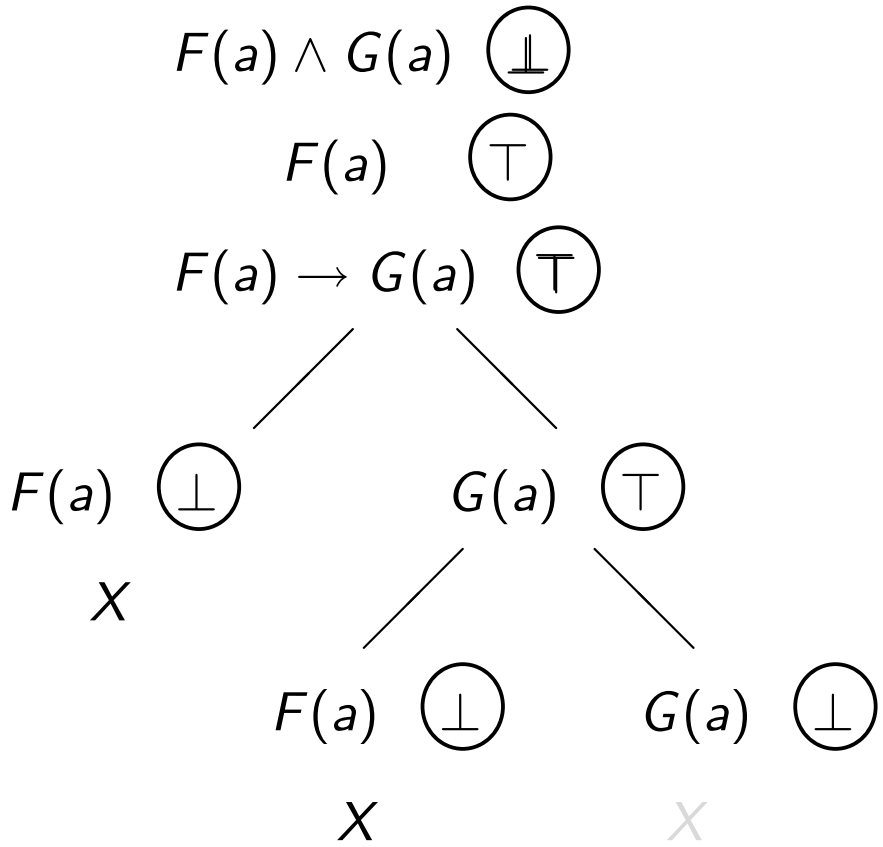
Sve grane su završile kontradikcijom pa zaključujemo da je dana formula valjana.



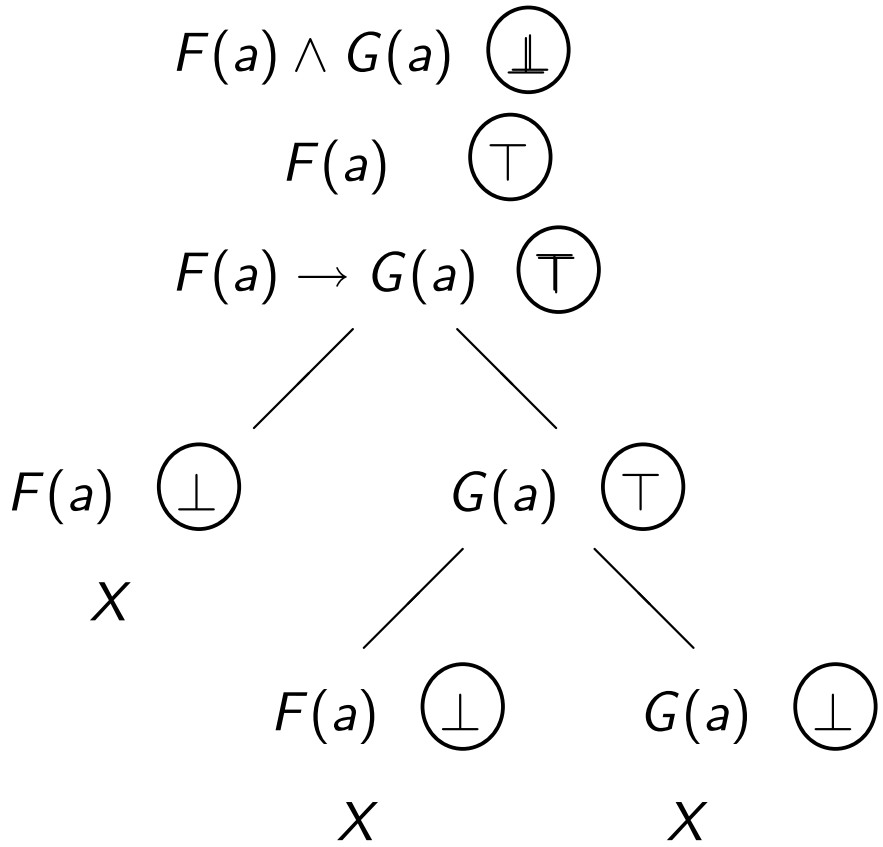
Sve grane su završile kontradikcijom pa zaključujemo da je dana formula valjana.



Sve grane su završile kontradikcijom pa zaključujemo da je dana formula valjana.



Sve grane su završile kontradikcijom pa zaključujemo da je dana formula valjana.



Sve grane su završile kontradikcijom pa zaključujemo da je dana formula valjana.

2. Pomoću glavnog testa ispitajte valjanost sljedećih formula:

   a) $(B \to (\forall x A(x) \wedge \forall x C(x))) \to (\neg B \vee \forall x(A(x) \wedge C(x)))$, pri čemu je B zatvorena formula;

   b) $\forall x \forall y P(x, y) \to (\exists y \exists x P(y, x) \ \vee \ \exists x \exists y P(x, y))$;

   c) $(\neg A \wedge (\exists x B(x) \vee \exists x C(x))) \ \leftrightarrow \ \neg(A \vee \forall x(\neg B(x) \wedge \neg C(x)))$;

   d) $\forall x \forall y (P(x, y) \wedge Q(x)) \to (\forall x \forall y P(x, y) \wedge \forall x Q(x))$.

Rješenje: Sve navedene formule su valjane.

3. Odredite prvo preneksnu normalnu formu formule

$$(\forall x F(x) \vee (\exists x F(x) \rightarrow \forall x G(x))) \rightarrow (\forall x F(x) \wedge \forall x G(x)),$$

a zatim ispitajte je li dobivena preneksna normalna forma oboriva.

4. Pomoću glavnog testa ispitajte vrijedi li:

a) $\{\forall x \forall y(F(x,y) \to \neg F(y,x))\} \models \neg \exists x F(x,x)$;

b) $\{\forall x(A(x) \to B(x))\} \models \forall y(\exists x(A(x) \wedge C(y,x)) \to \exists x(B(x) \wedge C(y,x)))$;

c) $\{\exists x \forall y B(x,y) \to A,\ \neg A \vee \exists x \exists y B(x,y)\} \models A \vee \neg \forall y \forall x B(y,x)$, gdje je formula A zatvorena;

d) $\forall x \forall y(R(x,y) \to \neg R(y,x)) \models \forall x \forall y(R(x,y) \to (R(y,x) \to R(x,x)))$;

e) $F \to \neg \forall y \exists x R(x,y),\ \forall x \exists y R(y,x) \vee \exists x R(x,x) \models \neg F \vee \forall x \forall y R(x,y)$;

f) $\exists x(R(x,x) \to \forall y R(x,y)) \models \forall x \forall y(\neg R(x,y) \to R(y,x))$. Ako tvrdnja ne vrijedi odredite barem jednu interpretaciju koja to dokazuje.

5. Ispitajte pomoću glavnog testa vrijedi li

$$F(a) \to G(b),\ \forall x(\neg F(x)) \models \neg G(b).$$

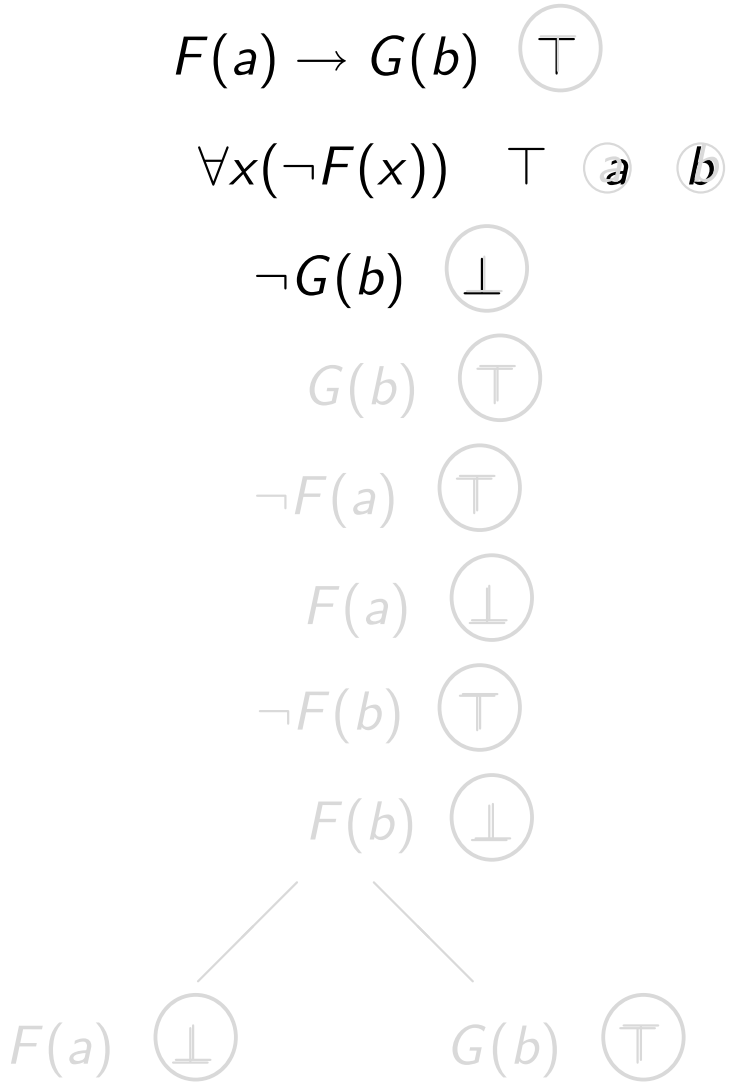
Rješenje: Pošto dana formula sadrži konstantske simbole a i b moramo prvo reći što raditi s njima prilikom glavnog testa.

Po definiciji strukture za svaki konstantski simbol mora postojati element u nosaču.

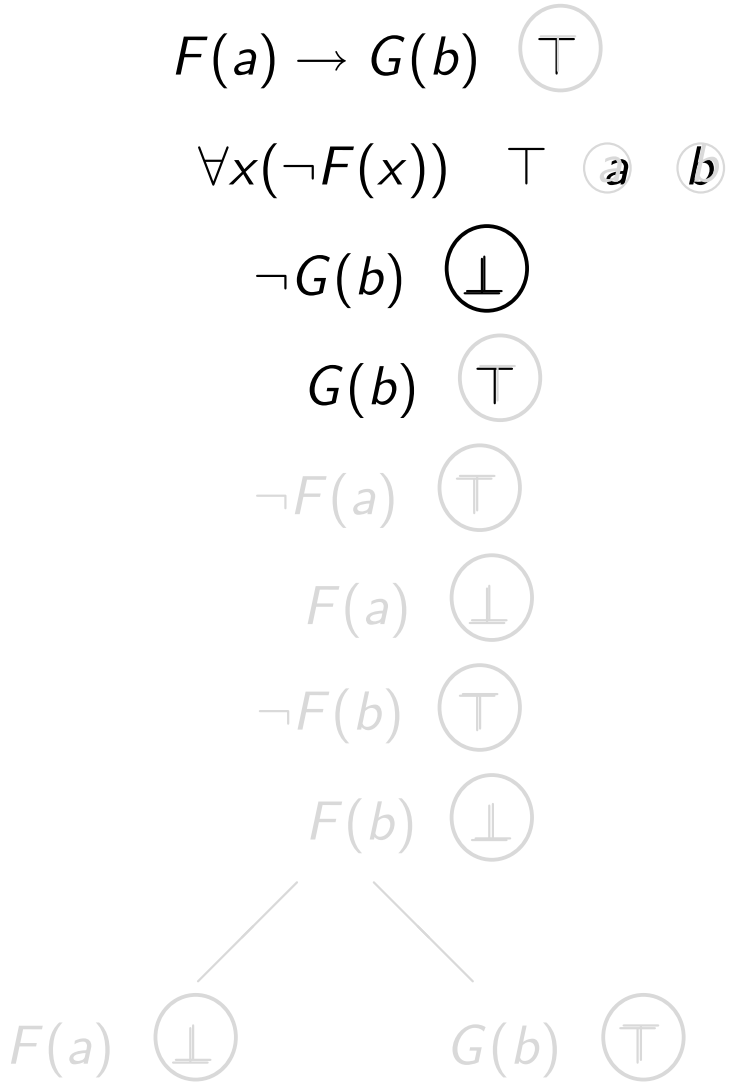
To znači da prije početka testiranja smatramo da nosač strukture sadrži barem dva elementa.

Interpretacije konstatskih simbola, kao i obično u ovoj točki, označavamo istim znakovima.

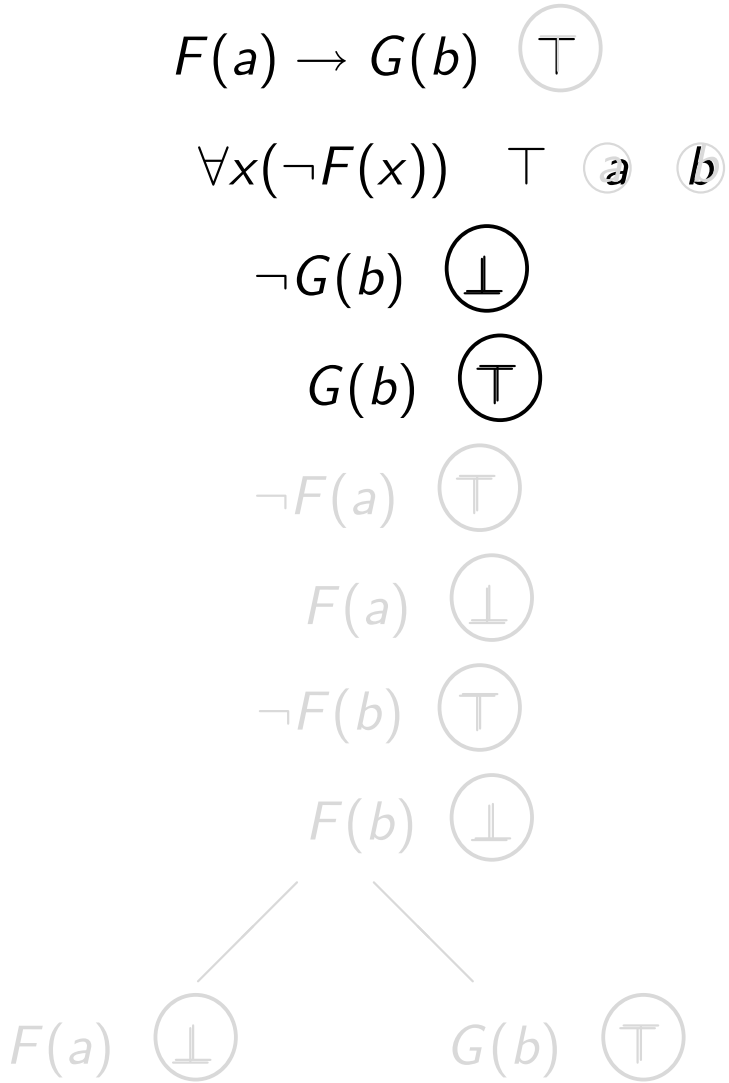
Glavni test zapisujemo u obliku stabla ovako:



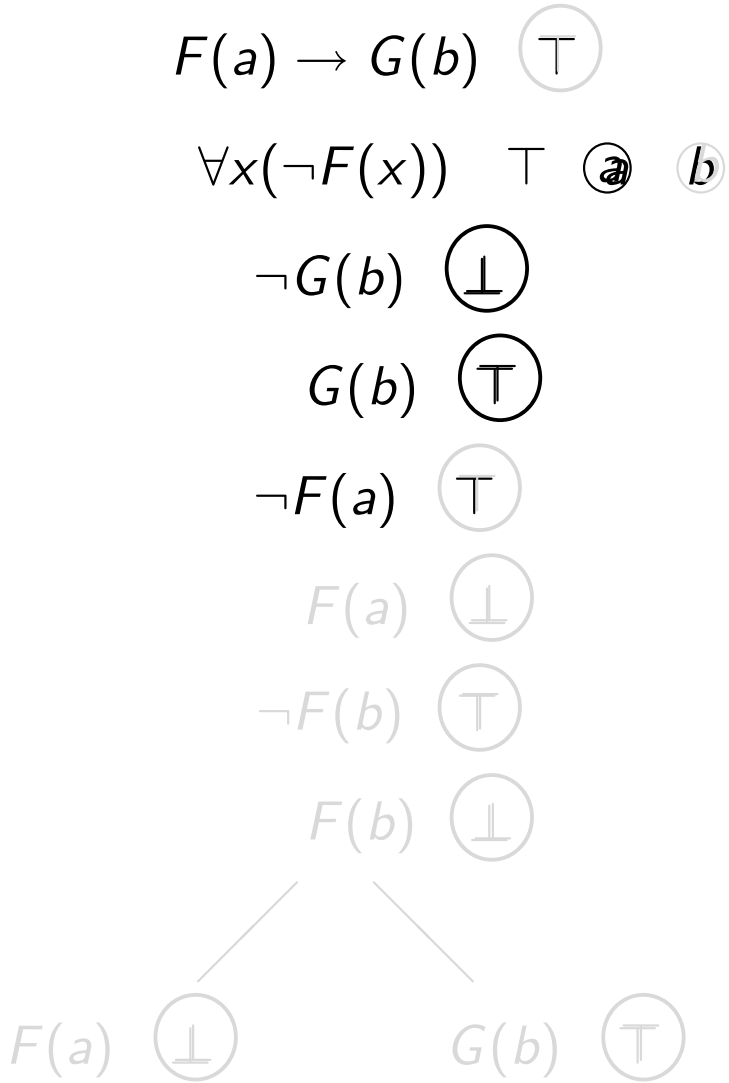
Glavni test zapisujemo u obliku stabla ovako:



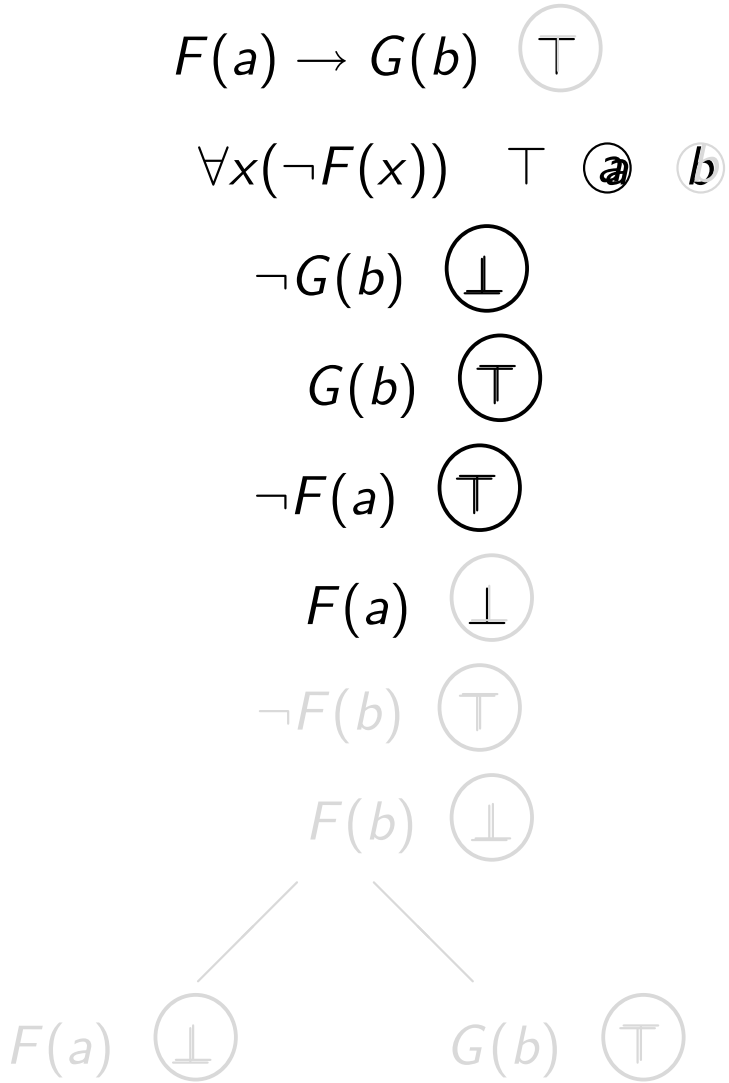
Glavni test zapisujemo u obliku stabla ovako:



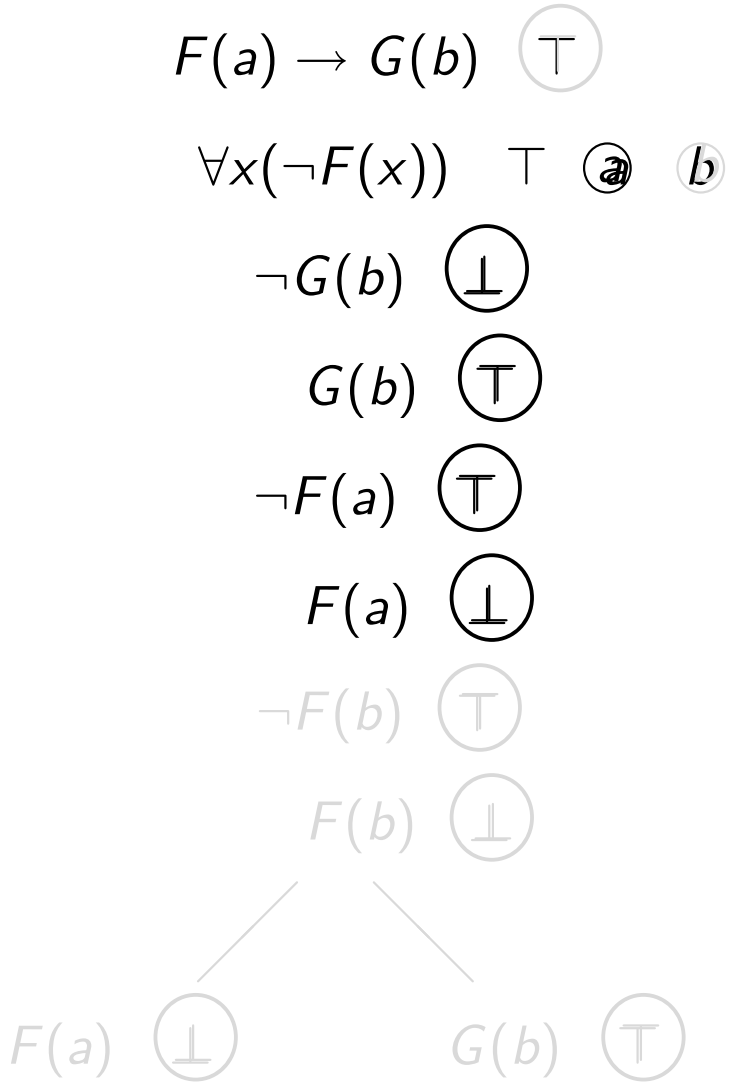
Glavni test zapisujemo u obliku stabla ovako:



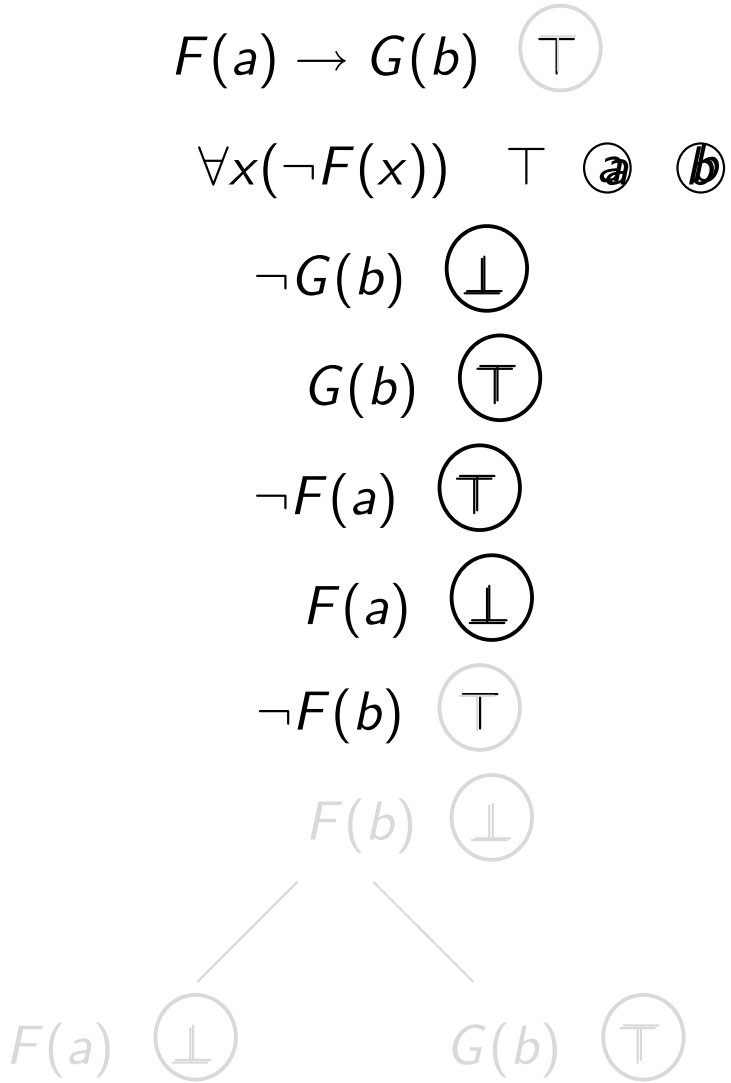
Glavni test zapisujemo u obliku stabla ovako:



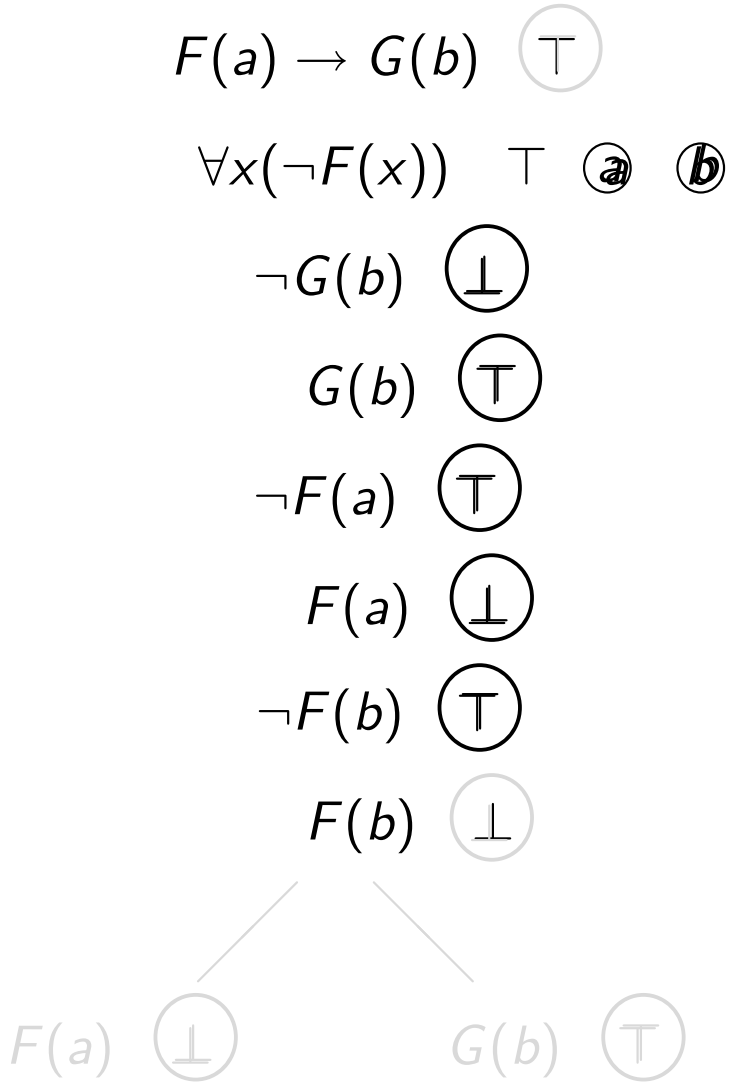
Glavni test zapisujemo u obliku stabla ovako:



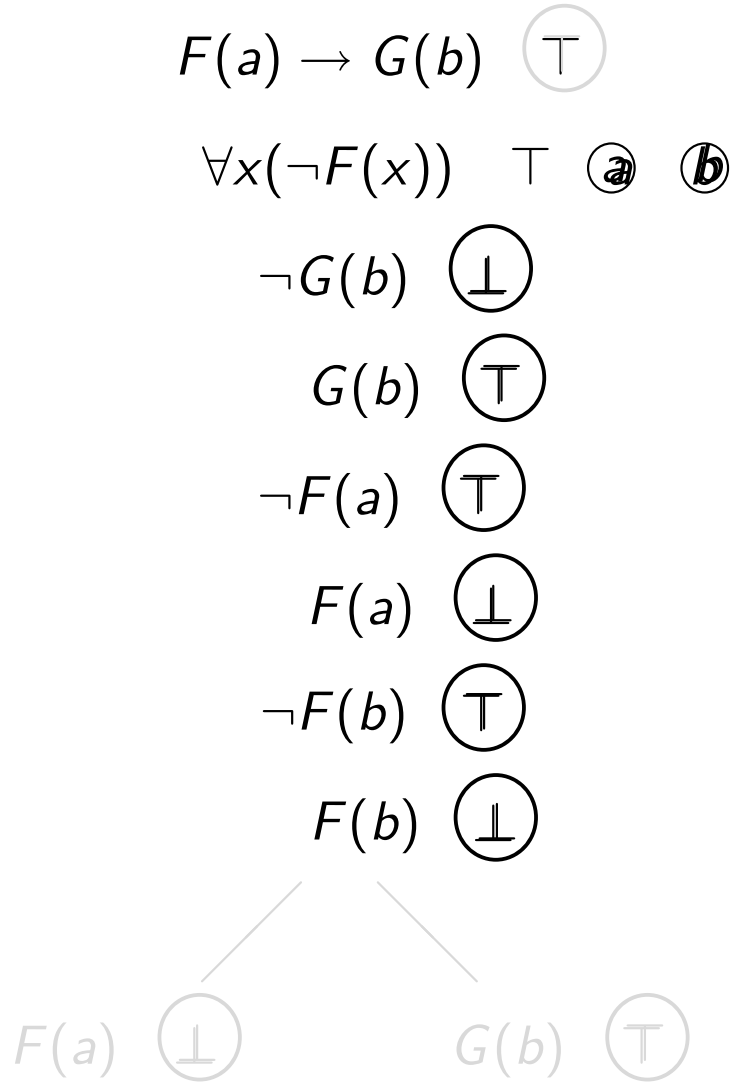
Glavni test zapisujemo u obliku stabla ovako:



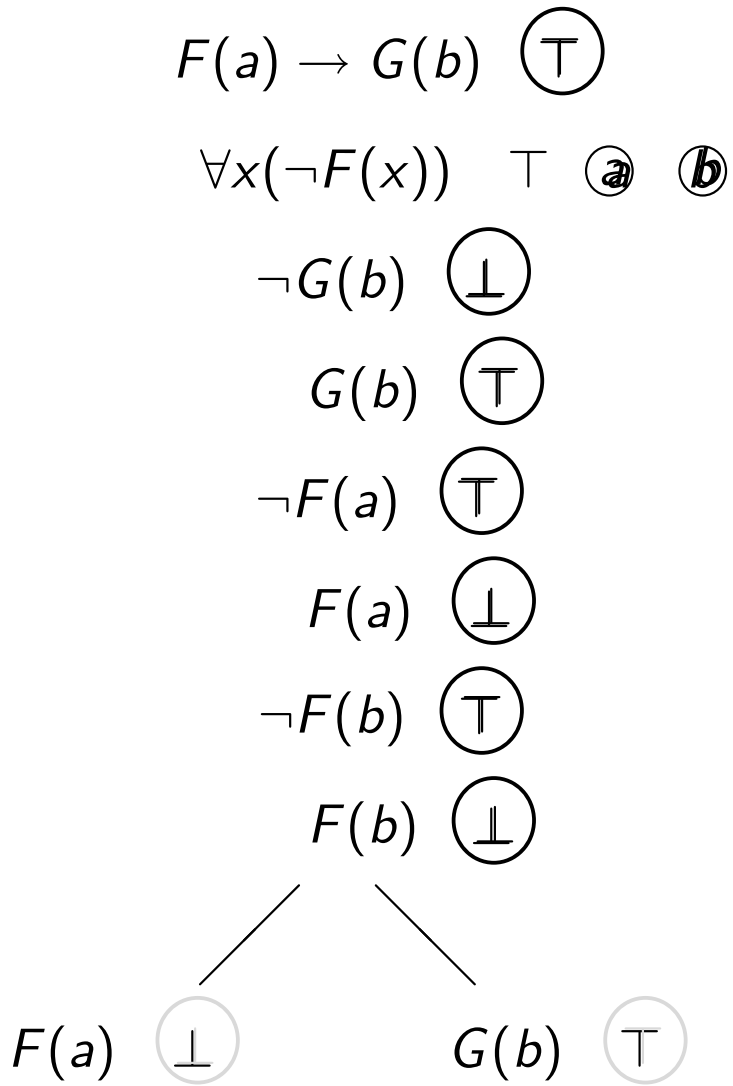
Glavni test zapisujemo u obliku stabla ovako:



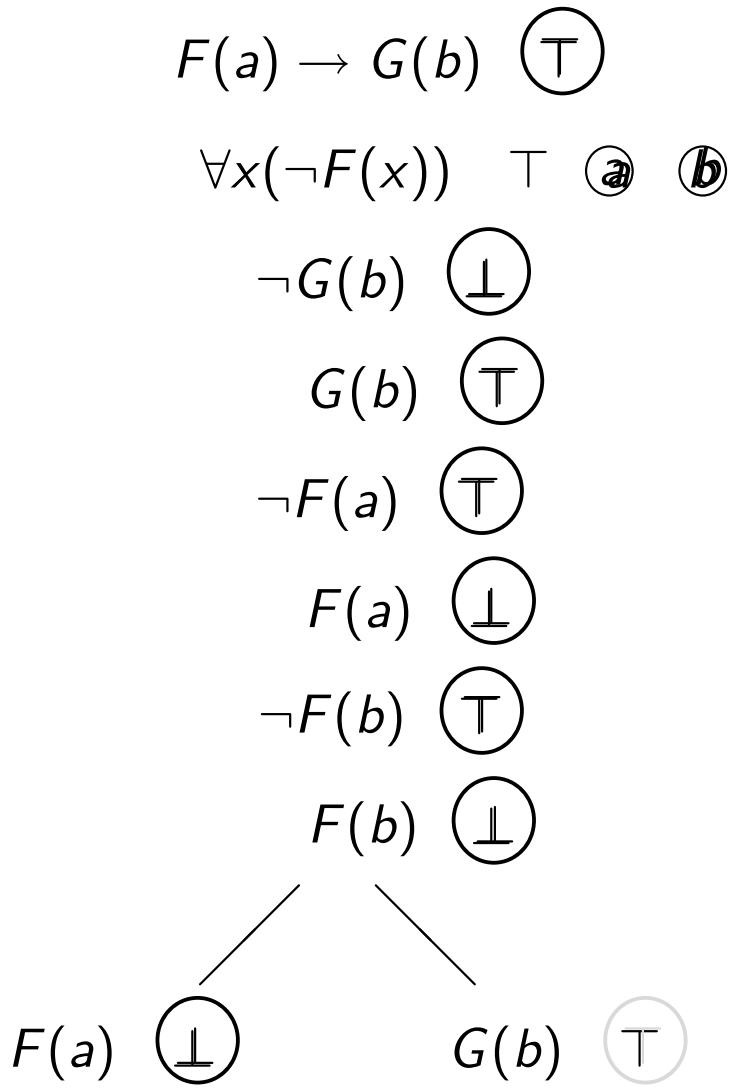
Glavni test zapisujemo u obliku stabla ovako:



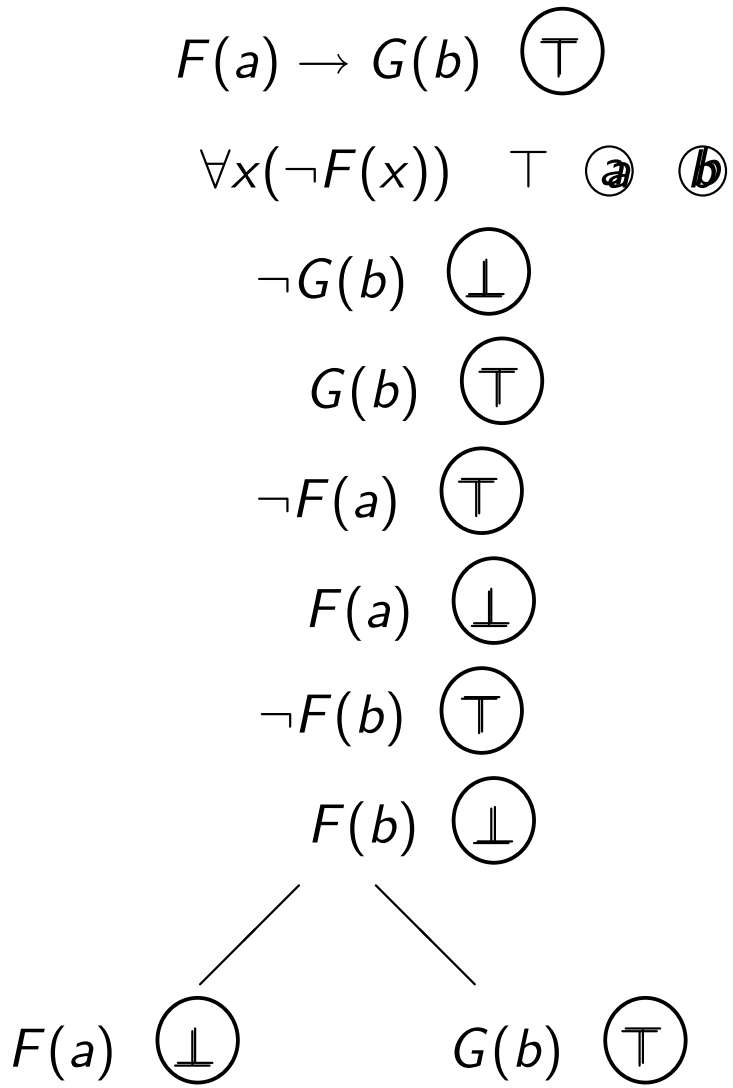
Glavni test zapisujemo u obliku stabla ovako:



Glavni test zapisujemo u obliku stabla ovako:



Glavni test zapisujemo u obliku stabla ovako:



Pošto nisu sve grane završile kontradikcijom (nije niti jedna!) zaključujemo da dana tvrdnja nije istinita, tj. formula $\neg G(b)$ logički ne slijedi iz skupa formula $\{F(a) \to G(b),\ \forall x(\neg F(x))\}$.

6. Pomoću glavnog testa ispitajte je li ispunjiva formula

$$(\forall x \exists y P(x,y) \wedge \forall x Q(x)) \wedge \neg \forall x \exists y (P(x,y) \wedge Q(x)).$$

7.

- Neka je A zatvorena formula, a B formula s točno jednom slobodnom varijablom.

  Koristeći glavni test dokažite ili opovrgnite

$$\{\exists x B(x) \to A,\ \neg A \vee \exists x B(x)\} \models A \vee \neg \exists x B(x).$$

8.

- Pomoću glavnog testa odredite barem dvije strukture koje dokazuju

$$\forall x(A(x) \to B(x)) \not\models \forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))).$$

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \; @_a \; b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \; \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \; \bot$

$\exists x(A(a) \wedge C(a,x)) \; \top \quad (..b..)$

$(**) \quad \exists x(B(a) \wedge C(x,x)) \quad \bot \; @_a \; b$

$A(a) \wedge C(a,b) \; \top$

$A(a) \; \top$

$C(a,b) \; \top$

Iz $(*)$ $\quad A(a) \to B(a) \; \top$

$A(a) \; \bot$ $\qquad$ $B(a) \; \top$

X $\qquad$ Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \; @_a \; b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \; \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \; \bot$

$\exists x(A(a) \wedge C(a,x)) \; \top \quad (..b..)$

$(**) \quad \exists x(B(a) \wedge C(x,x)) \quad \bot \; @_a \; b$

$A(a) \wedge C(a,b) \; \top$

$A(a) \; \top$

$C(a,b) \; \top$

Iz $(*)$ $\quad A(a) \to B(a) \; \top$

$A(a) \; \bot$ $\qquad$ $B(a) \; \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \; @_a \; b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \; \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \; \bot$

$\exists x(A(a) \wedge C(a,x)) \; \top \quad (..b..)$

$(**) \quad \exists x(B(a) \wedge C(x,x)) \quad \bot \; @_a \; b$

$A(a) \wedge C(a,b) \; \top$

$A(a) \; \top$

$C(a,b) \; \top$

Iz $(*)$ $\quad A(a) \to B(a) \; \top$

$A(a) \; \bot$ $\qquad$ $B(a) \; \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \; @_a \; b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**) \quad \exists x(B(a) \wedge C(x,x)) \quad \bot \; @_a \; b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot$ $\qquad$ $B(a) \quad \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \quad a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \quad a \quad b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot$ $\qquad$ $B(a) \quad \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \quad a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \quad a \quad b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot \qquad\qquad B(a) \quad \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \quad a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \quad a \quad b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot$ $\qquad$ $B(a) \quad \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \quad a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \quad a \quad b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot$ $\qquad$ $B(a) \quad \top$

X $\qquad$ Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \; a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \; \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \; \bot$

$\exists x(A(a) \wedge C(a,x)) \; \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \; a \quad b$

$A(a) \wedge C(a,b) \; \top$

$A(a) \; \top$

$C(a,b) \; \top$

Iz $(*)$ $\quad A(a) \to B(a) \; \top$

$A(a) \; \bot$ $\qquad$ $B(a) \; \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \; a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \; \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \; \bot$

$\exists x(A(a) \wedge C(a,x)) \; \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \; a \quad b$

$A(a) \wedge C(a,b) \; \top$

$A(a) \; \top$

$C(a,b) \; \top$

Iz $(*)$ $\quad A(a) \to B(a) \; \top$

$A(a) \; \bot$ $\qquad$ $B(a) \; \top$

X $\qquad$ Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \ @_a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \ @_a \quad b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot$ $\qquad$ $B(a) \quad \top$

X $\qquad$ Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \ @_a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \ @_a \quad b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot \qquad\qquad B(a) \quad \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \ @_a \quad b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \ @_a \quad b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

$C(a,b) \quad \top$

Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

$A(a) \quad \bot$ $\qquad$ $B(a) \quad \top$

X

Nastavak na sljedećem slajdu

$(*)$ $\quad \forall x(A(x) \to B(x)) \quad \top \ @ \ b$

$\forall y(\exists x(A(y) \wedge C(y,x)) \to \exists x(B(y) \wedge C(x,x))) \quad \bot \quad (..a..)$

$\exists x(A(a) \wedge C(a,x)) \to \exists x(B(a) \wedge C(x,x)) \quad \bot$

$\exists x(A(a) \wedge C(a,x)) \quad \top \quad (..b..)$

$(**)$ $\quad \exists x(B(a) \wedge C(x,x)) \quad \bot \ @ \ b$

$A(a) \wedge C(a,b) \quad \top$

$A(a) \quad \top$

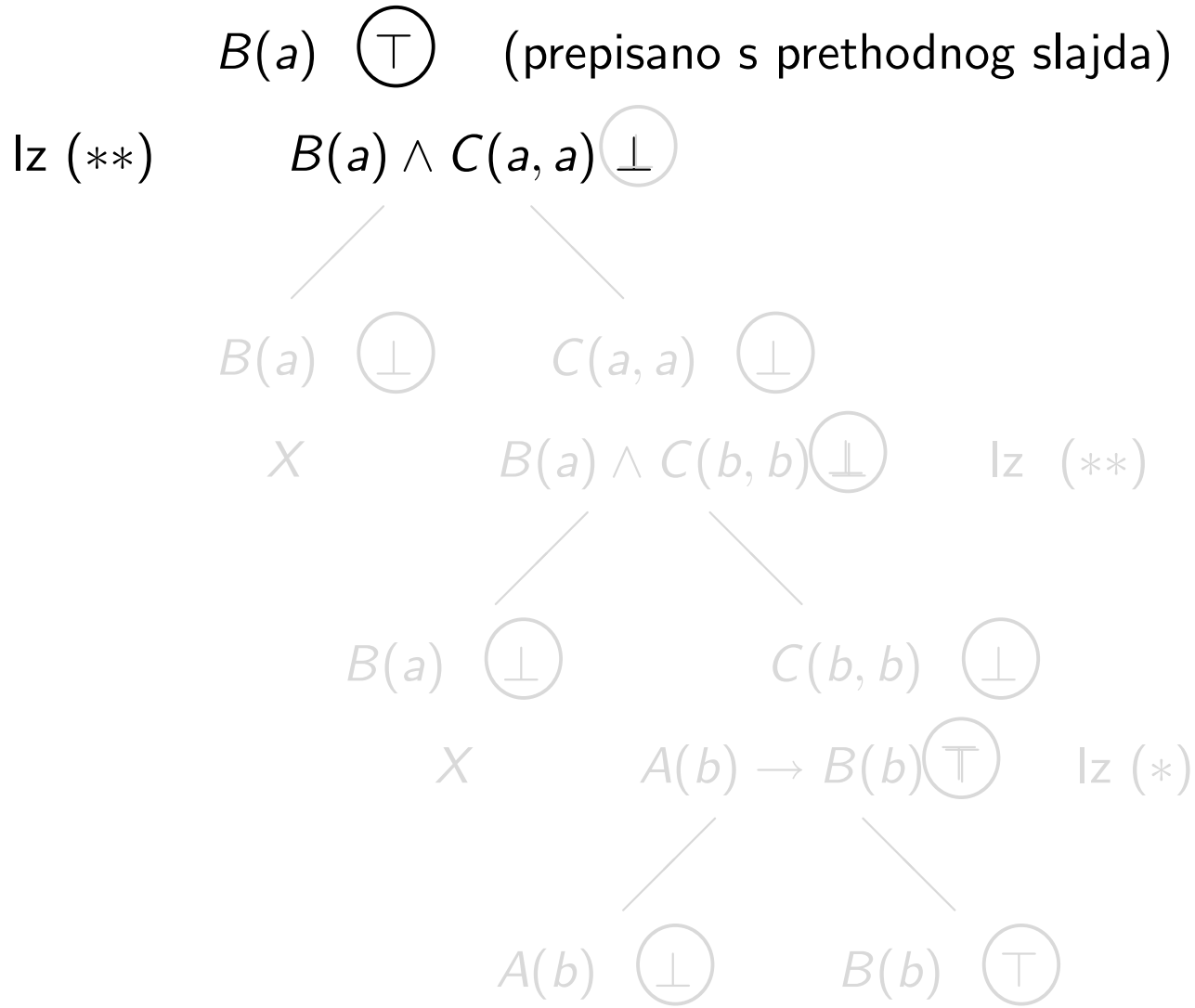
$C(a,b) \quad \top$

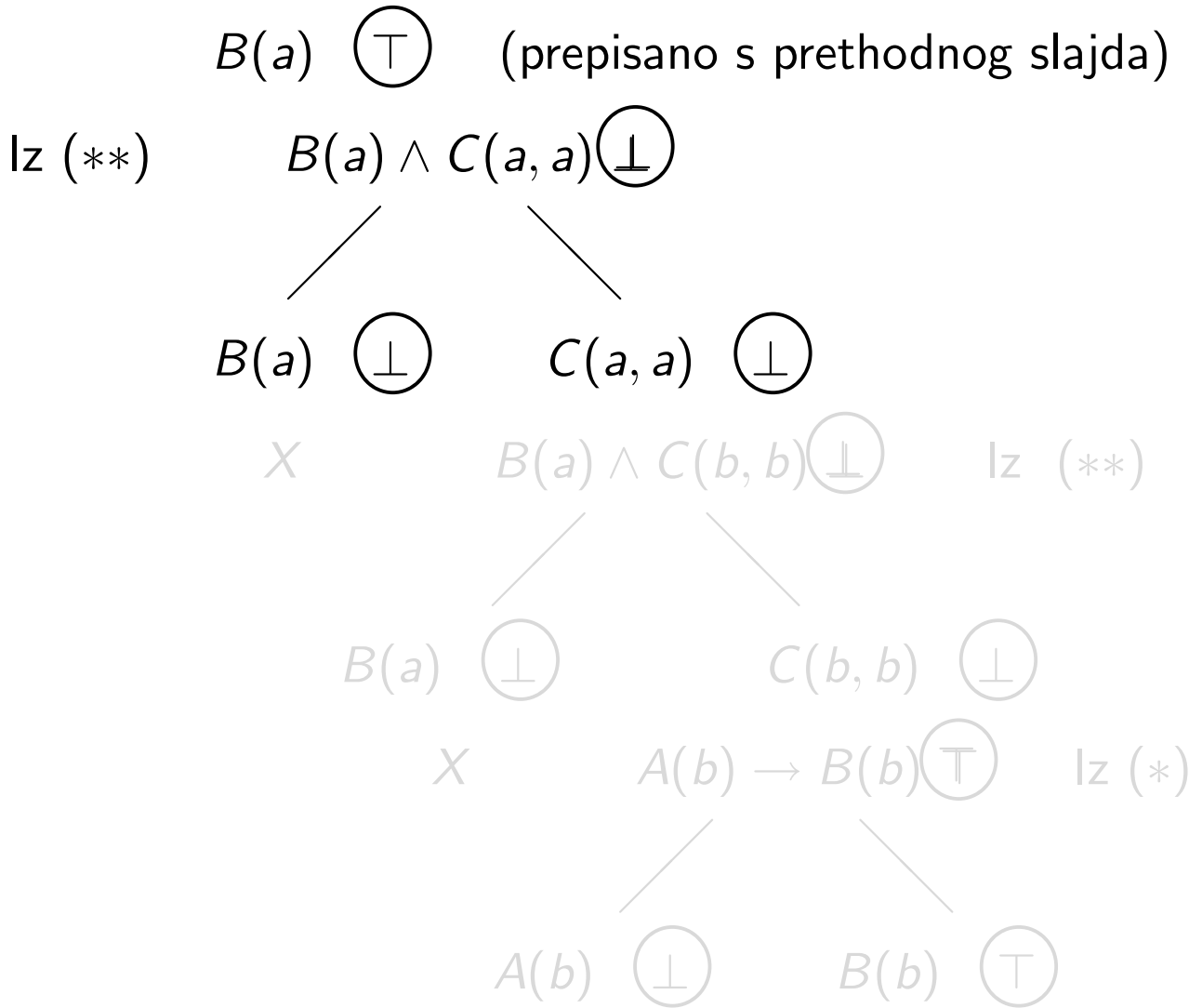
Iz $(*)$ $\quad A(a) \to B(a) \quad \top$

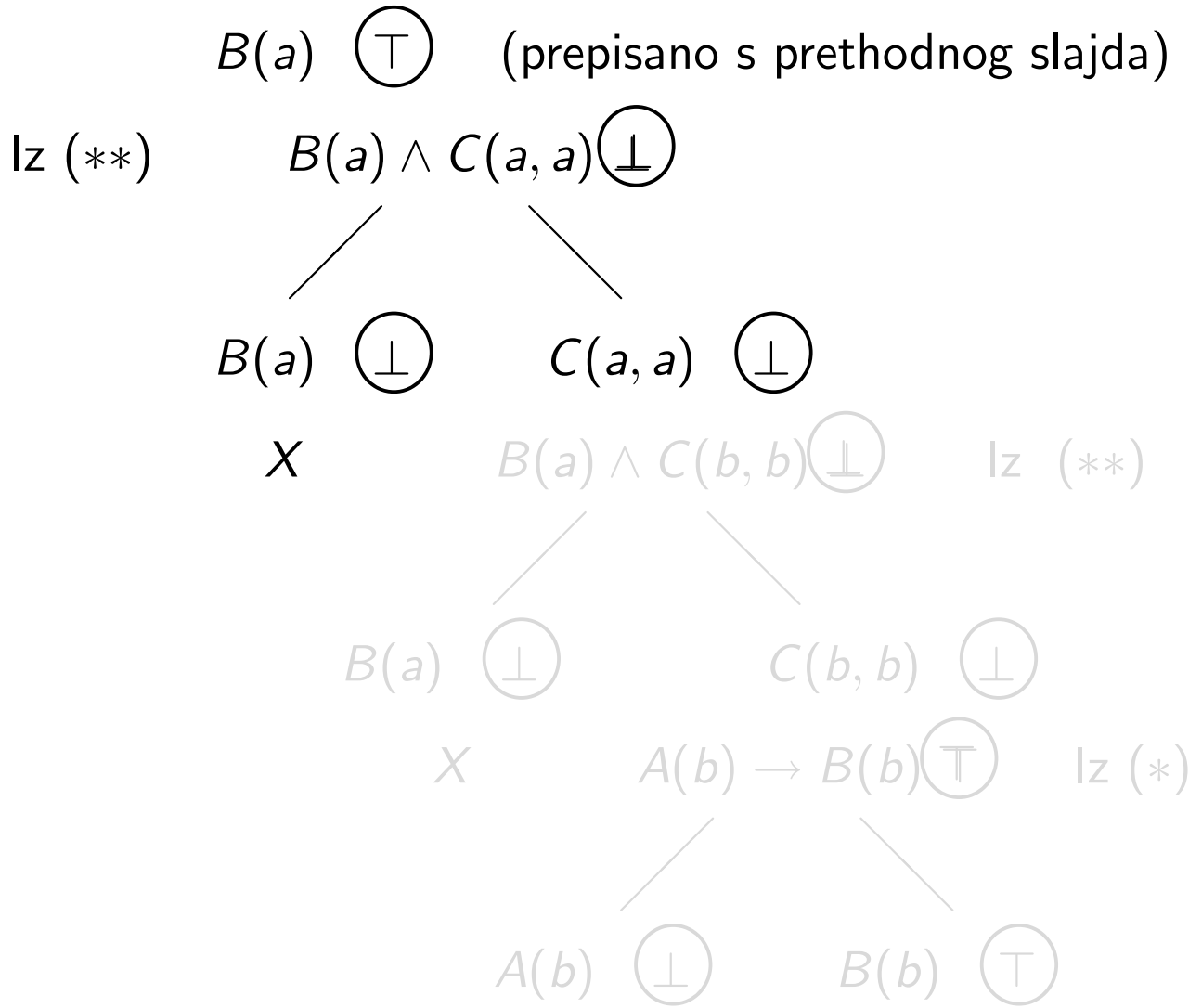
$A(a) \quad \bot$ $\qquad$ $B(a) \quad \top$

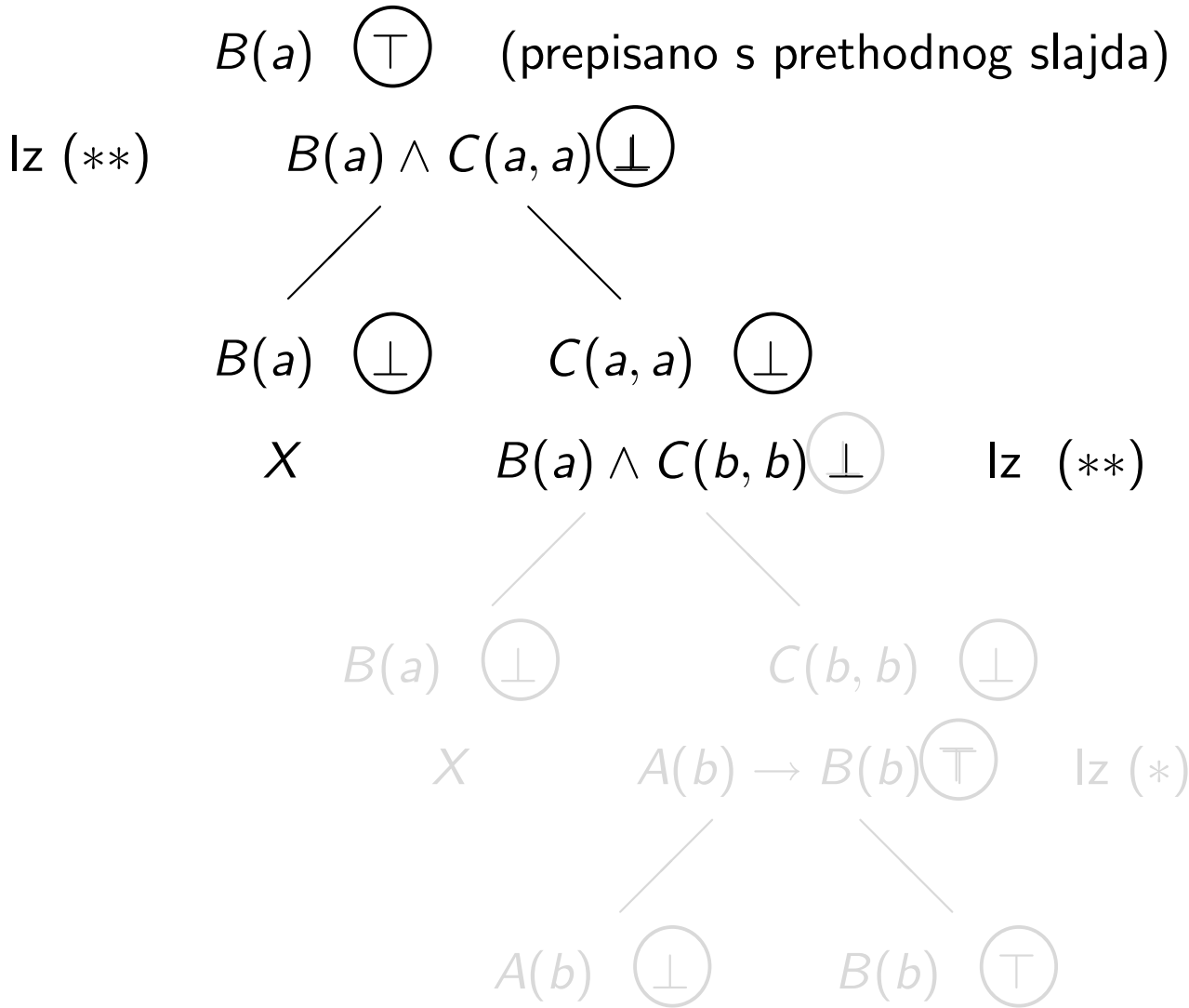
X $\qquad$ Nastavak na sljedećem slajdu

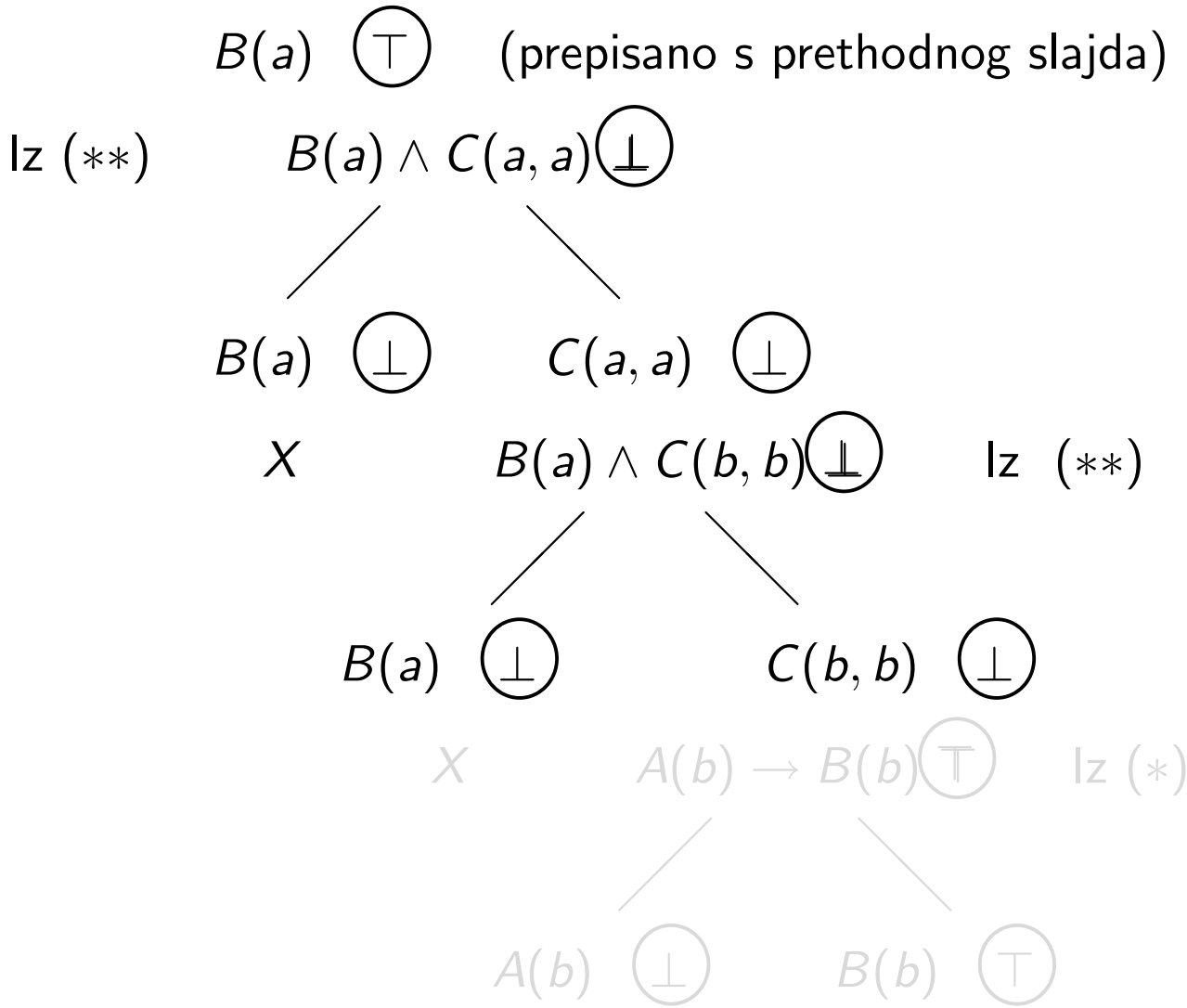
$B(a)$ $\top$ (prepisano s prethodnog slajda)

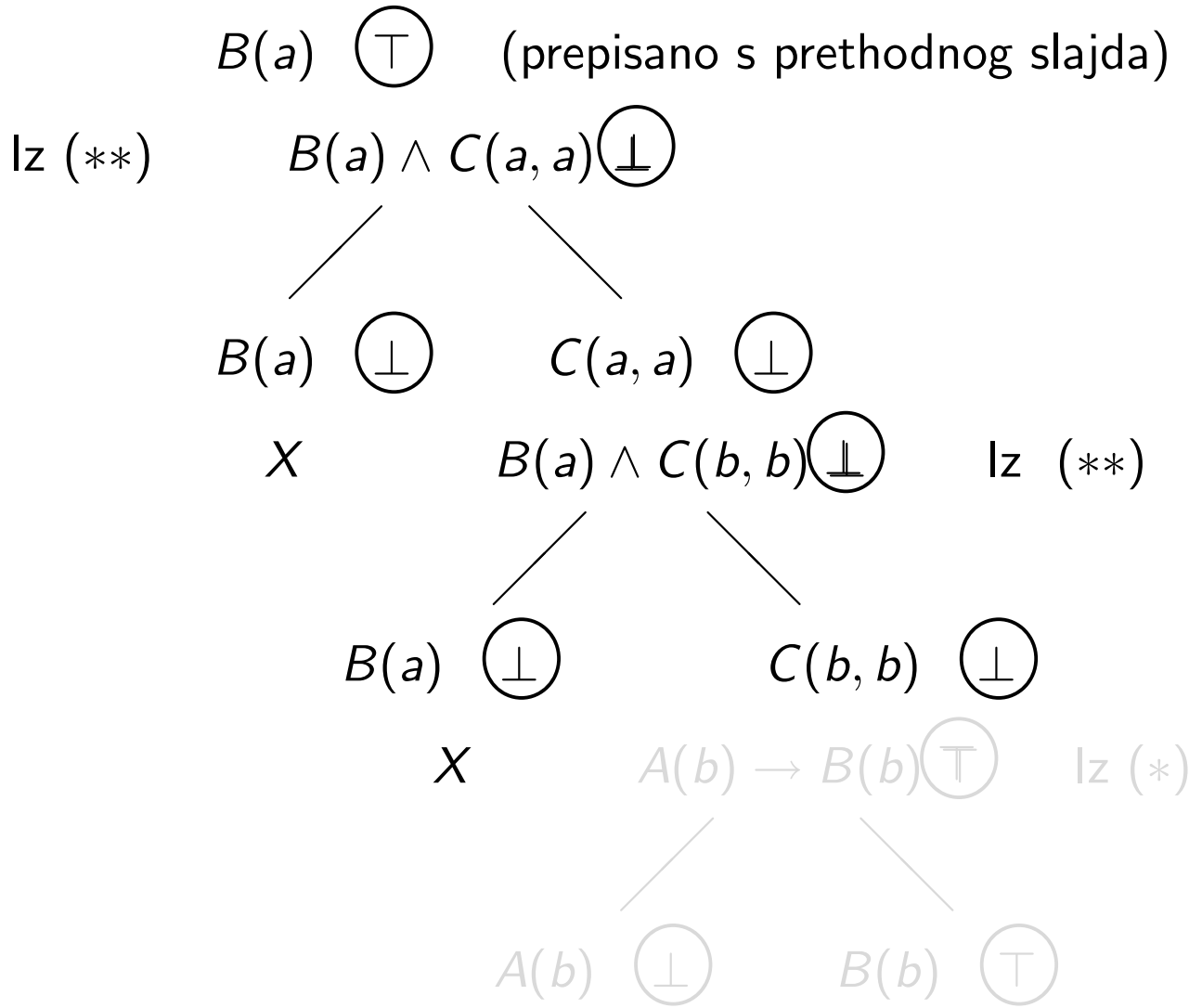

$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$ $B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$
Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$
Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$

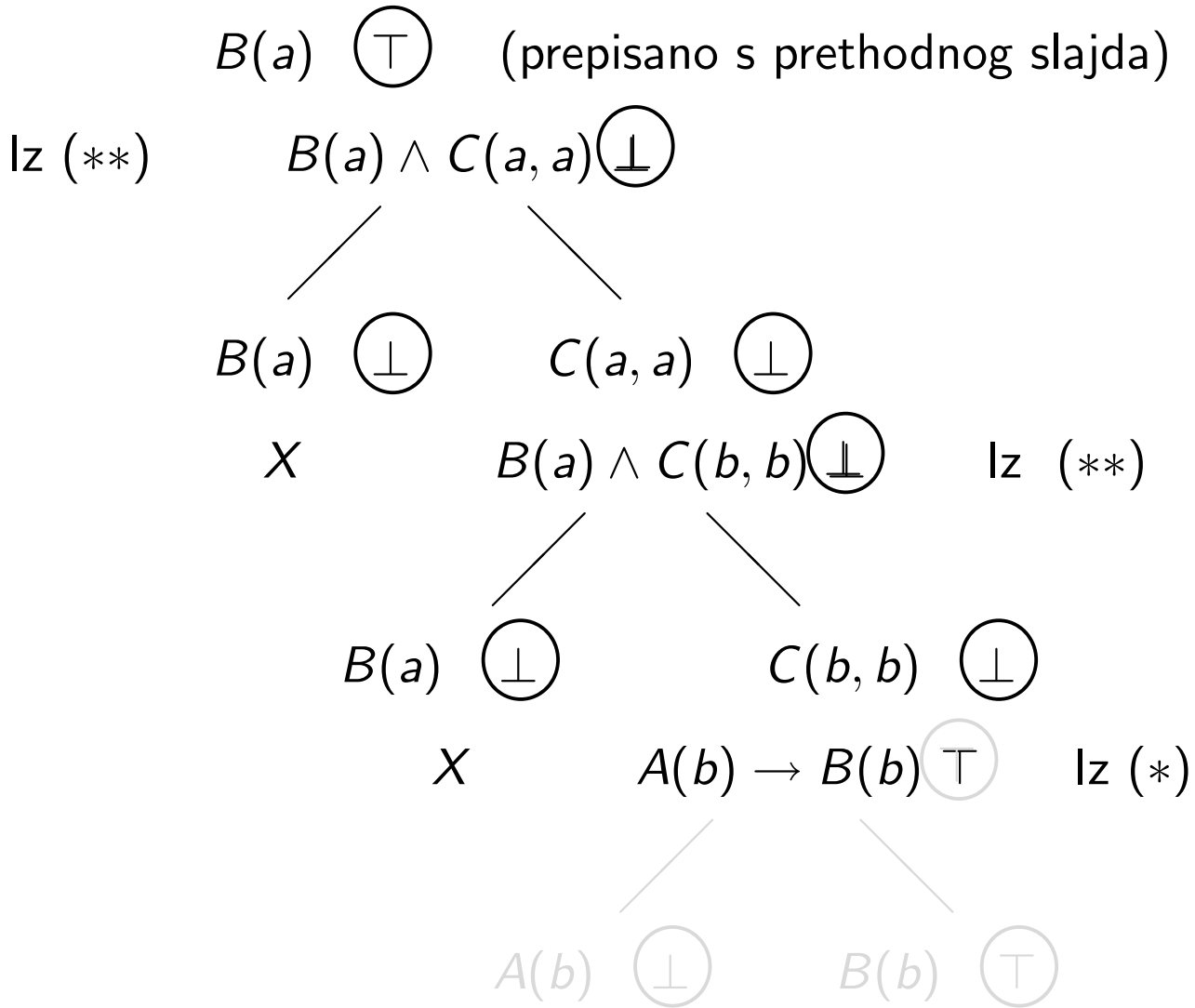

$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$
$B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$
Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$
Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$

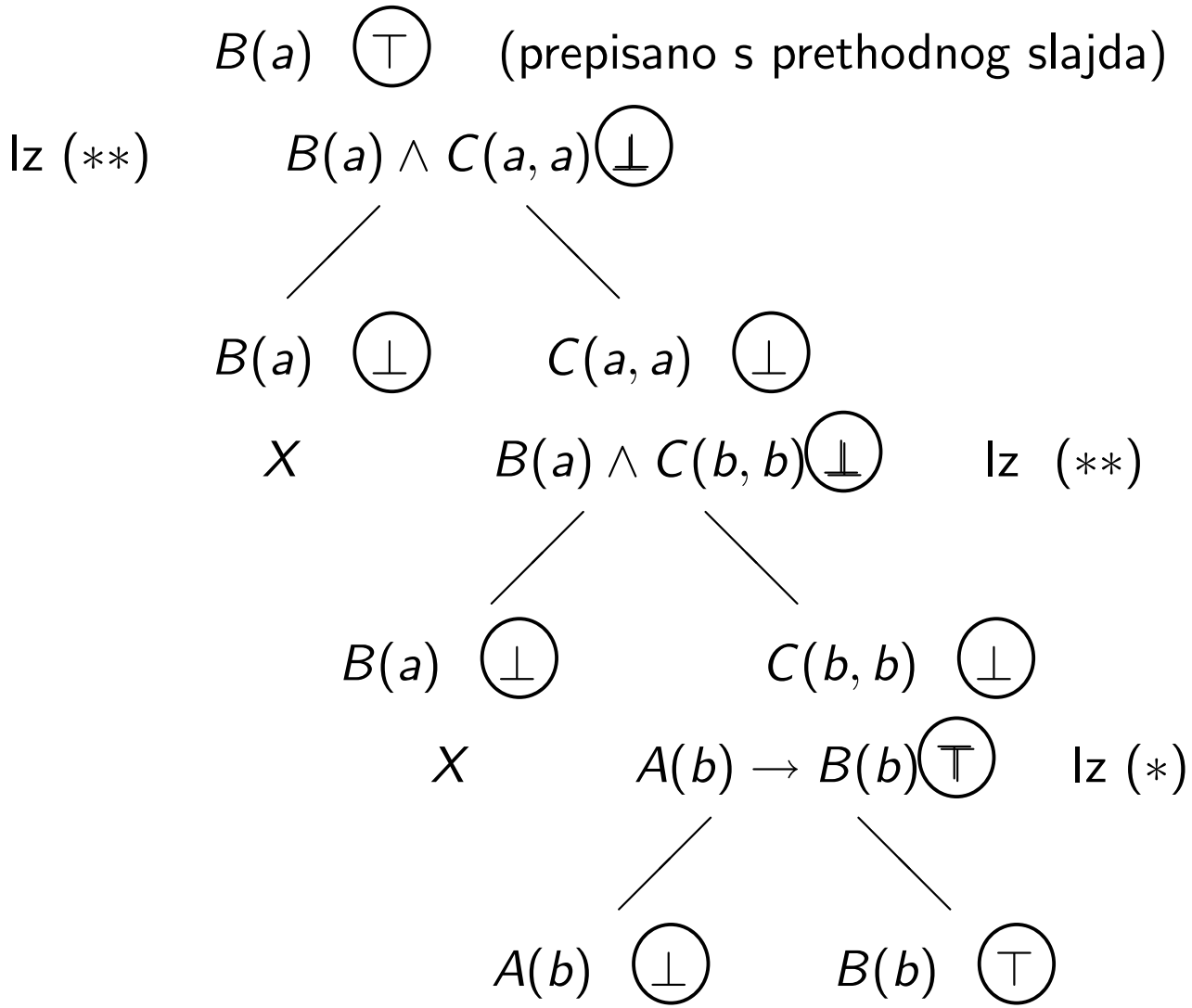

$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$
$B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$ Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$ Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$


$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$
$B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$
Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$
Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$


$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$
$B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$
Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$
Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$


$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$
$B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$
Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$
Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$


$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$
$B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$
Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$
Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$


$B(a)$ $\top$ (prepisano s prethodnog slajda)
Iz $(**)$
$B(a) \wedge C(a,a)$ $\bot$
$B(a)$ $\bot$
$C(a,a)$ $\bot$
X
$B(a) \wedge C(b,b)$ $\bot$
Iz $(**)$
$B(a)$ $\bot$
$C(b,b)$ $\bot$
X
$A(b) \to B(b)$ $\top$
Iz $(*)$
$A(b)$ $\bot$
$B(b)$ $\top$

Neka je $|\mathfrak{M}| = \{a, b\}$, te $A^{\mathfrak{M}} = \{a\}$, $B^{\mathfrak{M}} = \{a\}$, i $C^{\mathfrak{M}} = \{(a, b)\}$.

Zatim definiramo $|\mathfrak{N}| = \{a, b\}$, $A^{\mathfrak{N}} = \{a\}$, $B^{\mathfrak{N}} = \{a, b\}$ i $C^{\mathfrak{N}} = \{(a, b)\}$.

Tada su $\mathfrak{M}$ i $\mathfrak{N}$ tražene dvije strukture.

9.

- Prilikom ispitivanje valjanosti formule

$$\forall x \exists y A(x, y) \rightarrow \exists y \forall x A(x, y)$$

proveden je sljedeći glavni test:

$$\forall x \exists y A(x,y) \rightarrow \exists y \forall x A(x,y) \quad (\bot)$$

$$\forall x \exists y A(x,y)\ \top \quad (..a_1..) \quad a_1 \quad , \quad a_2$$

$$\exists y \forall x A(x,y)\ \bot \quad a_1 \quad , \quad a_2$$

$$\exists y A(a_1,y) \quad (\top) \quad (..a_2..)$$

$$A(a_1,a_2) \quad (\top)$$

$$\forall x A(x,a_1) \quad (\bot) \quad (!..a_1..)$$

$$A(a_1,a_1) \quad (\bot)$$

$$\exists y A(a_2,y) \quad (\top) \quad (!..a_1..)$$

$$A(a_2,a_1) \quad (\top)$$

$$\forall x A(x,a_2) \quad (\bot) \quad (!..a_2..)$$

$$A(a_2,a_2) \quad (\bot)$$

Uočite da je test završen, tj. provedena je analiza za sve formule i sve elemente.

Pošto test nije završen kontradikcijom možemo li zaključiti da dana formula nije valjana?

(Uočite da smo u primjeru 4 također ispitivali valjanost iste formule.

U gornjem testu smo s znakom ! označili da upotrebljavamo stari element, iako bismo po pravilu trebali uvoditi novi element.)

10. Neka je F zatvorena formula, a R dvomjesni relacijski simbol.

Koristeći glavni test dokažite ili opovrgnite

$$\{\neg F \vee \exists x \forall y(\neg R(x,y)),\ \forall y \exists x R(y,x) \vee \exists x R(x,x)\} \models F \to \exists x \exists y R(x,y).$$

Rješenje:

$$\neg F \vee \exists x \forall y(\neg R(x,y)) \quad (\top)$$

$$\forall y \exists x R(y,x) \vee \exists x R(x,x) \quad (\top)$$

$$F \to \exists x \exists y R(x,y) \quad (\bot)$$

$$F \quad (\top)$$

$$(I) \qquad \exists x \exists y R(x,y) \quad \bot \quad @$$

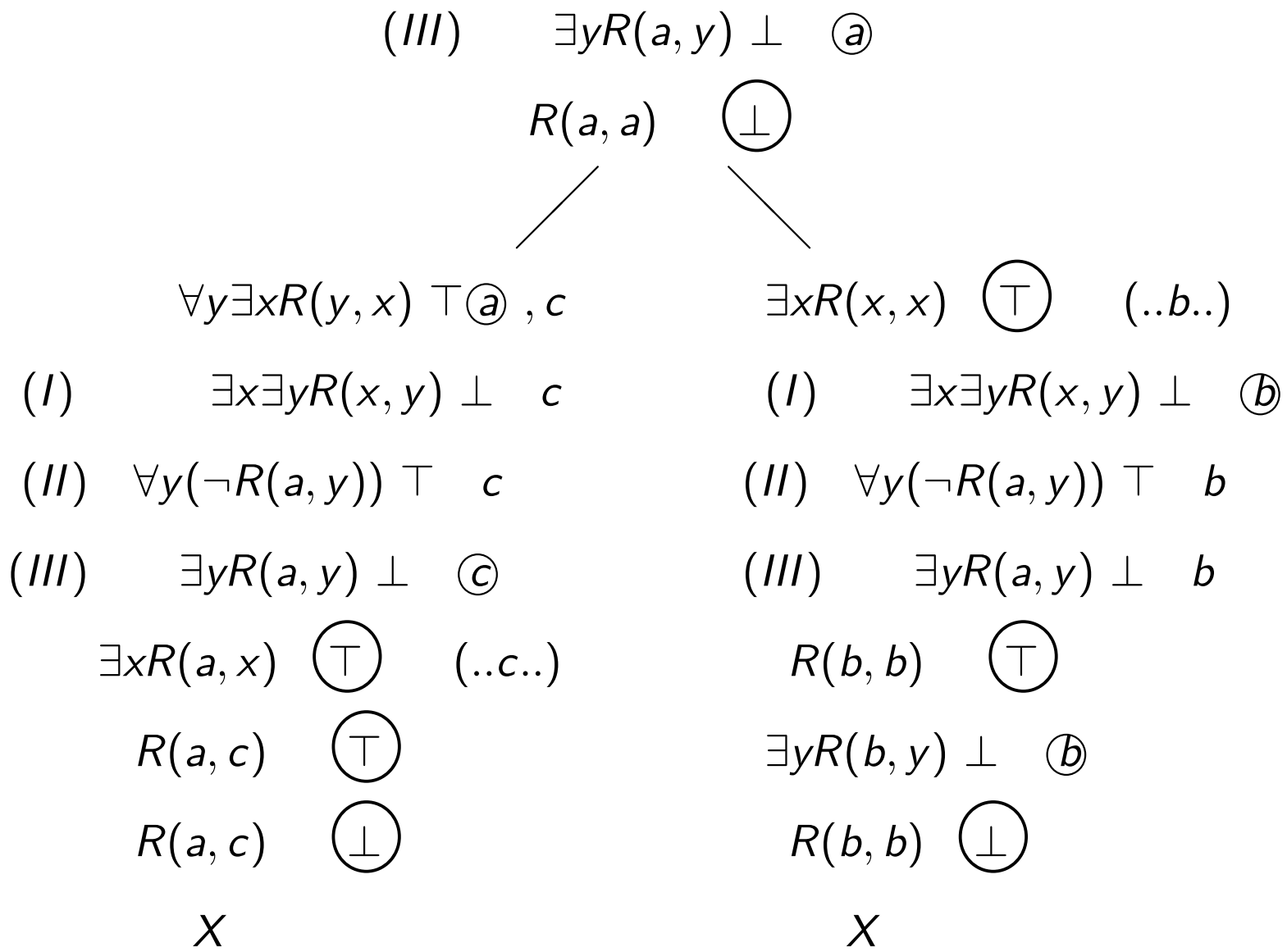
$$\neg F \quad (\top) \qquad\qquad \exists x \forall y(\neg R(x,y)) \quad (\top) \quad (..a..)$$

$$F \quad (\bot) \qquad\qquad (II) \quad \forall y(\neg R(a,y)) \quad \top \quad @$$

$$X \qquad\qquad \neg R(a,a) \quad (\top)$$

$$R(a,a) \quad (\bot)$$

$$(III) \qquad \exists y R(a,y) \ \bot \ @$$


(III) ∃yR(a, y) ⊥ ⓐ
R(a, a) ⊥
∀y∃xR(y, x) ⊤ⓐ , c
(I) ∃x∃yR(x, y) ⊥ c
(II) ∀y(¬R(a, y)) ⊤ c
(III) ∃yR(a, y) ⊥ ⓒ
∃xR(a, x) ⊤ (..c..)
R(a, c) ⊤
R(a, c) ⊥
X
∃xR(x, x) ⊤ (..b..)
(I) ∃x∃yR(x, y) ⊥ ⓑ
(II) ∀y(¬R(a, y)) ⊤ b
(III) ∃yR(a, y) ⊥ b
R(b, b) ⊤
∃yR(b, y) ⊥ ⓑ
R(b, b) ⊥
X

Pošto su sve grane završile kontradikcijom zaključujemo da je početna tvrdnja istinita.