Linear Algebra - Exercises

9 Linear Systems. Method of Elimination

1. Determine whether
a) $x = 1$, $y = -1$
b) $x = 2$, $y = -3$
are solutions to the linear equation

$$2x + 3y = -1.$$

Solution. a)

$$\begin{aligned} 2 \cdot 1 + 3 \cdot (-1) &= -1, \\ -1 &= -1, \end{aligned}$$

so $x = 1$, $y = -1$ is a solution.

b)

$$\begin{aligned} 2 \cdot 2 + 3 \cdot (-3) &= -1, \\ -5 &\neq -1, \end{aligned}$$

so $x = 2$, $y = -3$ is not a solution.

2.Determine if
a) $x = 0$, $y = 2$
b) $x = -1$, $y = -3$
are solutions to the system

$$\begin{array}{rcrcr} -x & + & 3y & = & -8, \\ 2x & + & y & = & -5. \end{array}$$

Solution. a) Substituting $x = 0$, $y = 2$ into Equation 1, we obtain

$$\begin{aligned} -1 \cdot 0 + 3 \cdot 2 &= -8, \\ 6 &\neq -8, \end{aligned}$$

so $x = 0$, $y = 2$ is not a solution.
b) Substituting $x = -1$, $y = -3$ into Equation 1, we obtain

$$\begin{aligned} -1 \cdot (-1) + 3 \cdot (-3) &= -8, \\ -8 &= -8, \end{aligned}$$

and into Equation 2, we obtain

$$\begin{aligned} 2 \cdot (-1) + (-3) &= -5, \\ -5 &= -5. \end{aligned}$$

Therefore, $x = -1$, $y = -3$ is a solution.

3. Solve the system

$$\begin{array}{rcrcr} 2x & + & y & = & 2, \\ 4x & - & y & = & 1. \end{array}$$

by the method of elimination.

Solution.

$\frac{1}{2} \cdot$ Equation 1

$$\begin{array}{rcrcr} x & + & \frac{1}{2}y & = & 1 \\ 4x & - & y & = & 1 \end{array}$$

Equation 2 $-$ 4 $\cdot$ Equation 1

$$\begin{array}{rcrcr} x & + & \frac{1}{2}y & = & 1 \\ & - & 3y & = & -3 \end{array}$$

>From Equation 2

$$y = 1$$

Substituting $y = 1$ into Equation 1

$$\begin{aligned} x + \frac{1}{2} &= 1 \\ x &= \frac{1}{2} \end{aligned}$$

The system has the unique solution

$$\begin{aligned} x &= \frac{1}{2} \\ y &= 1 \end{aligned}$$

4. Solve the system

$$\begin{array}{rcrcr} 2x & + & y & = & 1, \\ x & + & 2y & = & -1, \\ x & - & y & = & 2. \end{array}$$

by the method of elimination.

Solution.

Interchange Equation 1 and Equation 2

$$\begin{array}{rcrcr} x & + & 2y & = & -1 \\ 2x & + & y & = & 1 \\ x & - & y & = & 2 \end{array}$$

Equation 2 − 2 · Equation 1
Equation 3 − Equation 1

$$\begin{array}{rcrcr} x & + & 2y & = & -1 \\ & - & 3y & = & 3 \\ & - & 3y & = & 3 \end{array}$$

$-\frac{1}{3}$ · Equation 2
$-\frac{1}{3}$ · Equation 3

$$\begin{array}{rcrcr} x & + & 2y & = & -1 \\ & & y & = & -1 \\ & & y & = & -1 \end{array}$$

Equation 3 − Equation 2

$$\begin{array}{rcrcr} x & + & 2y & = & -1 \\ & & y & = & -1 \\ & & 0 & = & 0 \end{array}$$

We substitute $y = -1$ into Equation 1

$$x + 2 \cdot (-1) = -1$$
$$x = 1$$

The system has the unique solution

$$x = 1$$
$$y = -1$$

5. Solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1, \\ x & + & 2y & + & 2z & = & 1, \\ x & - & y & + & z & = & 3. \end{array}$$

by the method of elimination.

Solution.
Equation 2 $-$ Equation 1
Equation 3 $-$ Equation 1

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ & & y & + & z & = & 0 \\ & - & 2y & & & = & 3 \end{array}$$

Equation 3 $+ 2 \cdot$ Equation 2

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 1 \\ & & y & + & z & = & 0 \\ & & & & 2z & = & 2 \end{array}$$

>From Equation 3, we have

$$z = 1.$$

We substitute $z = 1$ into Equation 2:

$$\begin{aligned} y + 1 &= 0 \\ y &= -1. \end{aligned}$$

We substitute $y = -1$ and $z = 1$ into Equation 1:

$$\begin{aligned} x + (-1) + 1 &= 1 \\ x &= 1. \end{aligned}$$

The solution is

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 1 \end{aligned}$$

3. Solve the system

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 2, \\ x & + & 2y & + & z & = & 0, \\ x & + & y & & & = & 1. \end{array}$$

by the method of elimination.

10 Matrices

1. Let

$$A = \begin{bmatrix} 2 & 5 & -5 \\ 2 & -2 & 3 \end{bmatrix}.$$

Write
a) the size of A **b)** $Row_2(A)$ **c)** $Col_1(A)$ **d)** $(2,3)$ entry of A

Solution.
a) The size of A is 2×3.
b)

$$Row_2(A) = \begin{bmatrix} 2 & -2 & 3 \end{bmatrix}$$

c)

$$Col_1(A) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

d) $a_{23} = 3$

2. Determine which of the following matrices are square matrices. For square matrices, write the order and the diagonal.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ -4 & 4 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -4 \\ 0 & 1 & 0 \end{bmatrix}.$$

Solution. A is not a square matrix.
B is a square matrix of order 2, the diagonal of B is

$$2,\ 4$$

C is a square matrix of order 3, the diagonal of C is

$$1,\ 1,\ 0$$

2. Let

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 5 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}.$$

Find, if possible

a) $A + B$ **b)** $A + C$ **c)** $2C + 3I_2$

Solution.
a) A and B are of the same size 2×3, so $A + B$ is defined

$$A + B = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

b) $A + C$ is not defined because A and C are not of the same size.
c)

$$2C + 3I_2 = 2 \cdot \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ 2 & -1 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \\ 3 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & -1 & 3 & 1 \end{bmatrix}.$$

Find, if possible

a) A^T **b)** $A + B$ **c)** $A^T + B$

Solution.
a)

$$A^T = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 0 & 1 \end{bmatrix}$$

b) $A + B$ is not defined
c)

$$A^T + B = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 3 & -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 5 & -1 \\ 5 & -3 & 3 & 2 \end{bmatrix}$$

10.1 Homework Problems

1. Let

$$A = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Write
a) the size of A **b)** $Row_3(A)$ **c)** $Col_4(A)$ **d)** $(2, 4)$ entry of A

2. Let

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -2 & 1 \\ 5 & -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 7 \\ -1 & -7 \\ 10 & 5 \end{bmatrix}.$$

Compute, if possible

a) $A - B$ **b)** $A + C$ **c)** $A - B^T$ **d)** $A^T + C$

3. Let

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 6 & 0 \\ -1 & -6 & 6 \end{bmatrix}.$$

Compute

a) $A + 4B$ **b)** $2A - 3I$ **c)** $-\frac{1}{2}C$ **d)** $C + 3I$

10.2 Answers to Homework Problems

1. a) 5×4 **b)** $\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 3 \end{bmatrix}$ **d)** 1

2. a) Not defined. **b)** $\begin{bmatrix} 3 & 6 \\ 0 & -4 \\ 10 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 2 & -6 \\ 3 & 4 \\ -1 & -4 \end{bmatrix}$ **d)** Not defined.

3. a) $\begin{bmatrix} 1 & 9 \\ -2 & -4 \end{bmatrix}$ **b)** $\begin{bmatrix} -1 & 10 \\ -4 & 5 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 & -1 & -2 \\ -\frac{1}{2} & -3 & 0 \\ \frac{1}{2} & 3 & -3 \end{bmatrix}$ **d)** $\begin{bmatrix} 3 & 2 & 4 \\ 1 & 9 & 0 \\ -1 & -6 & 9 \end{bmatrix}$

11 Matrix Multiplication

1. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Find, if possible

a) AB **b)** AC

Solution. **a)**

$$AB = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 + 0 + 0 = 2$$

b) AC is not defined because the number of columns of A is not equal to the number of rows of C.

2. Let

$$A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -2 & 3 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Find, if possible

a) AB **b)** AC

Solution. **a)**

$$\begin{array}{cccc} A & B & = & AB \\ 4 \times 2 & 4 \times 2 & & not\,defined \end{array}$$

The product AB is not defined because the sizes do not match (the number of columns of A is not equal to the number of rows of B).

b)

$$\begin{array}{cccc} A & C & = & AC \\ 4 \times 2 & 2 \times 3 & & 4 \times 3 \end{array}$$

Therefore, AC is a 4×3 matrix. Write AC as a blank 4×3 matrix and compute, one by one, the elements of AC:

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 11 & 3 & 1 \\ -3 & -1 & 0 \\ 4 & 2 & -1 \\ 6 & 0 & 3 \end{bmatrix}$$

3. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}.$$

Determine whether $AB = BA$.

Solution.

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -1 & -3 \end{bmatrix}$$

Therefore, $AB \neq BA$.

4. Let

$$A = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

Find, if possible

a) $A^2 = AA$ **b)** $A^T A$ **c)** AA^T

Solution. **a)**

$$\begin{matrix} A & A & = & AA \\ 1 \times 3 & 1 \times 3 & & not\, defined \end{matrix}$$

The product AA is not defined.

b)

$$A^T = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

We have

$$\begin{matrix} A & A^T & = & AA^T \\ 1 \times 3 & 3 \times 1 & & 1 \times 1 \end{matrix}$$

so AA^T is defined,

$$AA^T = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = [5]$$

c)

$$\begin{matrix} A^T & A & = & AA^T \\ 3 \times 1 & 1 \times 3 & & 3 \times 3 \end{matrix}$$

$$A^T A = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

11.1 Homework Problems

1.

$$A = \begin{bmatrix} 0 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Find, if possible

a) AB **b)** AC

2. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 3 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Find, if possible

a) AB **b)** AC

3. Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 2 \\ -1 & -1 & 3 \end{bmatrix}.$$

Compute

a) AB **b)** A^T **c)** B^T **d)** $B^T A^T$

e) Check that $(AB)^T = B^T A^T$.

11.2 Answers to Homework Problems

1. a) Not defined. **b)** 1.

2. a) $\begin{bmatrix} -1 & 0 \\ 5 & -3 \\ 4 & -3 \end{bmatrix}$ **b)** Not defined.

3. a) $\begin{bmatrix} -1 & -1 & 1 \\ -2 & -2 & 6 \end{bmatrix}$ **b)** $\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 & -1 \\ 0 & -1 \\ 2 & 3 \end{bmatrix}$ **d)** $\begin{bmatrix} -1 & -2 \\ -1 & -2 \\ 1 & 6 \end{bmatrix}$

12 Solutions of Linear Systems of Equations

1. Write the augmented matrix representing the linear system

$$\begin{array}{rcrcrcl} x & - & 4y & + & z & = & -2, \\ 2x & + & 3y & & & = & -1. \end{array}$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & -4 & 1 & -2 \\ 2 & 3 & 0 & -1 \end{array}\right]$$

2. The following matrix represents a linear system in variables x, y and z.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 3 & 2 \\ 1 & 0 & -1 & -3 \\ -1 & 2 & 5 & 1 \end{array}\right]$$

Write the system.

Solution.

$$\begin{array}{rcrcrcl} 2x & + & y & - & z & = & 0 \\ & & y & + & 3z & = & 2 \\ x & & & - & z & = & -3 \\ -x & + & 2y & + & 5z & = & 1 \end{array}$$

3. Determine which of the following matrices are in reduced row echelon form:

$$A = \left[\begin{array}{ccc|c} -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A is not in reduced row echelon form because the pivot in Row_1 is not equal to 1.

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

B is in reduced row echelon form.

$$C = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 3 & 1 \end{array}\right]$$

C is not in reduced row echelon form because the pivot in Row_3 is not to the right of the pivot in Row_2.

$$D = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

D is not in reduced row echelon form because not all the entries above the pivot in Row_2 are zero.

$$E = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

E is in reduced row echelon form.

4. The following augmented matrices represent systems of linear equations in variables x, y and z. In each case either state the general solution or that no solution exists.

a) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ **b)** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & -5 \\ 0 & 0 & 0 & 3 \end{array}\right]$ **c)** $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solution. **a)** The system corresponding to the matrix is

$$\begin{array}{rcrcl} x & & + & 2z & = & 4 \\ & y & + & 3z & = & 3 \\ & & & 0 & = & 0 \end{array}$$

The system has infinitely many solutions. The general solution is

$$x = 4 - 2z$$
$$y = 3 - 3z$$

where z is any real number.

b) The equation corresponding to the third row is

$$0 = 3.$$

Therefore, the system has no solution.

c) The system has the unique solution

$$x = -1$$
$$y = 2$$
$$z = 6$$

5. Solve the system

$$\begin{array}{rcrcl} 3x & + & 2y & = & 1, \\ x & + & y & = & 1. \end{array}$$

by Gauss-Jordan reduction.

Solution. The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 1 & 1 \end{array}\right]$$

We do Gauss-Jordan reduction to transform the matrix in reduced row echelon form.

$$\begin{array}{r} Row_2 \\ Row_1 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & 1 \end{array}\right]$$

$$Row_2 - 3Row_1 \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -2 \end{array}\right]$$

$$-Row_2 \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

$$Row_1 - Row_2 \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

The system has the unique solution

$$x = -1, \quad y = 2.$$

6. Solve the system

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 2 \\ 2x & - & y & + & z & = & 1 \\ 3x & & & + & 2z & = & 3 \end{array}$$

by Gauss-Jordan reduction.
Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & 0 & 2 & 3 \end{array}\right]$$

$$\begin{array}{r} \\ Row_2 - 2Row_1 \\ Row_3 - 3Row_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -3 & -1 & -3 \\ 0 & -3 & -1 & -3 \end{array}\right]$$

$$\begin{array}{r} \\ -\frac{1}{3}Row_2 \\ Row_3 - Row_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$Row_1 - Row_2 \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 1 \\ 0 & 1 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We read

$$x + \frac{2}{3}z = 1$$
$$y + \frac{1}{3}z = 1$$

The system has infinitely many solutions. The general solution is

$$x = 1 - \frac{2}{3}z$$
$$y = 1 - \frac{1}{3}z$$

where z is any real number.

12.1 Homework Problems

1. Determine which of the following matrices are in reduced row echelon form:

$$A = \left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 0 & 1 & 5 & 7 \end{array}\right], \quad B = \left[\begin{array}{ccc|c} 0 & 1 & 2 & 5 \\ 1 & 0 & 3 & 0 \end{array}\right], \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. The following augmented matrices represent systems of linear equations in variables x, y and z. In each case either state the general solution or that no solution exists.

$$\textbf{a)} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \textbf{b)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \textbf{c)} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3. Solve the system

$$\begin{array}{rcrcr} 2x & + & 3y & = & 0 \\ x & + & 2y & = & -1 \end{array}$$

by Gauss-Jordan reduction.

4. Solve the system

$$\begin{array}{rcrcr} x & - & y & = & 6 \\ 2x & - & 3z & = & 16 \\ 2y & + & 7z & = & 4 \end{array}$$

by Gauss-Jordan reduction.

5. Solve the system

$$\begin{array}{rcrcrcr} x & - & y & - & z & = & 1 \\ -x & + & 2y & - & 3z & = & -4 \\ 3x & - & 2y & - & 7z & = & 0 \end{array}$$

by Gauss-Jordan reduction.
6. Solve the system

$$\begin{array}{rcrcrcl} 2x & - & y & + & z & = & 1 \\ 4x & - & 2y & + & 2z & = & 1 \end{array}$$

by Gauss-Jordan reduction.

12.2 Answers to Homework Problems

1. A is in reduced row echelon form, B, C and D are not in reduced row echelon form.
2. a) No solution.
b) The unique solution

$$x = 0, \quad y = -5, \quad z = 2$$

c) Infinitely many solutions. The general solution is

$$\begin{aligned} x &= -2z - 1 \\ y &= 2z + 2 \end{aligned}$$

where z is any real number.
3. The unique solution

$$x = 3, \quad y = -2.$$

4. The unique solution

$$x = 8, \quad y = 2, \quad z = 0.$$

5. Infinitely many solutions. The general solution is

$$\begin{aligned} x &= 5z - 2 \\ y &= 4z - 3 \end{aligned}$$

where z is any real number.
6. No solution.

13 The Inverse of a Matrix

1. Let

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}.$$

Show that

$$B = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -2 \end{bmatrix}$$

is the inverse of A.

Solution. We have to show that

$$AB = BA = I.$$

We multiply AB and obtain

$$AB = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly,

$$BA = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Find the inverse of the matrix A, if it exists.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Solution. We form the matrix

$$[A|I] = \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

and perform Gauss-Jordan reduction.

$$\frac{1}{2}Row_1 \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$$Row_2 - Row_1 \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

3. Find the inverse of the matrix A, if it exists.

$$A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

Solution.

$$[A|I] = \left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} Row_2 \\ Row_1 \end{array} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ Row_2 - 2Row_1 \end{array} \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

The matrix has no inverse.

4. Find the inverse of the matrix A, if it exists.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ Row_3 - Row_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ Row_3 \\ Row_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{array}{r} \\ \\ Row_3 - 2Row_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & -3 & 2 & 1 & -2 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -\frac{1}{3}Row_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

$$\begin{array}{r} \\ Row_2 - 2Row_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{array}\right]$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

13.1 Homework Problems

For each of given matrices, find the inverse, if it exists.

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ **2.** $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ **3.** $C = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

13.2 Answers to Homework Problems

1. A has no inverse.

2. $B^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ **3.** $C^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$

14 Determinants

1. Let
$$A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}.$$
Compute $det(A)$.

Solution.
$$det(A) = 2 \cdot (-2) - 1 \cdot 3 = -4 - 3 = -7.$$

2. Compute
$$\begin{vmatrix} 5 & 1 \\ -4 & 0 \end{vmatrix}.$$

Solution.
$$\begin{vmatrix} 5 & 1 \\ -4 & 0 \end{vmatrix} = 5 \cdot 0 - 1 \cdot (-4) = 0 + 4 = 4.$$

3. Compute
$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 3 & 0 & 1 \end{vmatrix}.$$

Solution.
$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ 3 & 0 & 1 \end{vmatrix}\begin{matrix} 1 & 1 \\ 2 & 0 \\ 3 & 0 \end{matrix} = \begin{matrix} 1 \cdot 0 \cdot 1 + 1 \cdot (-1) \cdot 3 + 2 \cdot 2 \cdot 0 \\ -2 \cdot 0 \cdot 3 - 1 \cdot (-1) \cdot 0 - 1 \cdot 2 \cdot 1 \end{matrix}$$
$$= 0 - 3 + 0 - 0 - 0 - 2 = -5$$

4. Compute
$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 4 \end{vmatrix}.$$

Solution.
$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ 2 & 0 & 4 \end{vmatrix}\begin{matrix} 2 & 1 \\ 0 & 1 \\ 2 & 0 \end{matrix} = \begin{matrix} 2 \cdot 1 \cdot 4 + 1 \cdot 3 \cdot 2 + 0 \cdot 0 \cdot 0 \\ -0 \cdot 1 \cdot 2 - 2 \cdot 3 \cdot 0 - 1 \cdot 0 \cdot 4 \end{matrix}$$
$$= 8 + 6 = 14$$

14.1 Homework Problems

1. Let

$$A = \begin{bmatrix} 5 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}.$$

Compute $det(A)$, $det(B)$.

2. Compute

a) $\begin{vmatrix} -\frac{1}{2} & 3 \\ 2 & -2 \end{vmatrix}$ **b)** $\begin{vmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 4 & 3 & 2 \end{vmatrix}$

14.2 Answers to Homework Problems

1. $det(A) = -9$, $det(B) = 6$.
2. **a)** -5. **b)** 60.

15 Cofactor Expansion and Applications

1. Let

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

Compute cofactors A_{12} and A_{31}.

Solution. To compute A_{12}, we first have to find the matrix M_{12}. It is the matrix obtained by deleting the first row and the second column of A,

$$M_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 2 \end{bmatrix}.$$

Now,

$$A_{12} = (-1)^{1+2} det(M_{12}) = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = -(4 \cdot 2 - 6 \cdot 7) = -(8 - 42) = 34.$$

Similarly,

$$A_{31} = (-1)^{3+1} det(M_{31}) = \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -16.$$

2. Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

Compute $det(A)$ using cofactor expansion about the first row.

Solution.

$$\begin{aligned} det(A) &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \\ &= (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix} + (-1)^{2+1} 2 \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} + 0 \\ &= -1 + (-2) = -3 \end{aligned}$$

3. Evaluate the determinant

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & -2 & 3 \end{vmatrix}.$$

Solution. For cofactor expansion, we select the row or the column with maximal number of zeros. Expanding about the third row, we obtain

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & -2 & 3 \end{vmatrix} = (-1)^{3+1} 3 \begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix}$$

Expanding about the first column,

$$= 3\left((-1)^{1+1} 2 \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} + (-1)^{2+1} 2 \begin{vmatrix} -3 & 4 \\ -2 & 3 \end{vmatrix}\right) = 3(2 \cdot 9 + (-1) \cdot 2 \cdot (-1)) = 60$$

Remark. The determinant

$$\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix}$$

can also be computed using the formula for 3×3 determinant.

4. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

a) Write the adjoint matrix of A.
b) Find the inverse of A.

Solution. **a)**

$$adj A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

b)

$$det(A) = -2$$

$$A^{-1} = \frac{1}{det(A)} adj A = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

5. Using properties of determinants, compute the determinants of the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 4 & 0 \\ 1 & 2 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 12 & 0 \\ 3 & 4 & 0 \\ 5 & 5 & 0 \end{bmatrix}.$$

Solution. A is a diagonal matrix, so

$$det(A) = 1 \cdot (-4) \cdot (-3) = 12.$$

B has two equal rows, so
$$det(B) = 0.$$
C has a zero column, so
$$det(C) = 0.$$

6. Evaluate
$$\begin{vmatrix} 3 & 3 & 3 \\ 5 & 1 & 1 \\ 3 & 4 & 3 \end{vmatrix}$$
using properties of determinants.

Solution.
$$\begin{vmatrix} 3 & 3 & 3 \\ 5 & 1 & 1 \\ 3 & 4 & 3 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 1 & 1 \\ 3 & 4 & 3 \end{vmatrix}$$
$Row_2 - Row_1$
$$= 3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 0 \\ 3 & 4 & 3 \end{vmatrix}$$
expansion about the second row
$$= 3(-1)^{2+1}4 \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 3(-4)(-1) = 12$$

7. Let
$$A = \begin{bmatrix} 1 & 0 & 0 \\ x & y & z \\ 1 & 2 & 3 \end{bmatrix}$$
Suppose that we know
$$det(A) = 5.$$
Evaluate the following determinants:
$$\textbf{a)} \begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix}, \quad \textbf{b)} \begin{vmatrix} 1 & 0 & 0 \\ x & y & z \\ 5 & 10 & 15 \end{vmatrix}, \quad \textbf{c)} \begin{vmatrix} 1 & 0 & 0 \\ x+1 & y+2 & z+3 \\ 1 & 2 & 3 \end{vmatrix}.$$

Solution. **a)** The determinant is obtained from $det(A)$ by interchanging two rows, so
$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = -det(A) = -5.$$

b) The determinant is obtained from $det(A)$ by multiplying the third row by 5 and therefore

$$\begin{vmatrix} 1 & 0 & 0 \\ x & y & z \\ 5 & 10 & 15 \end{vmatrix} = 5det(A) = 25.$$

b) The determinant is obtained from $det(A)$ by adding Row_3 to Row_2, so

$$\begin{vmatrix} 1 & 0 & 0 \\ x+1 & y+2 & z+3 \\ 1 & 2 & 3 \end{vmatrix} = det(A) = 5$$

15.1 Homework Problems

1. Let

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -1 & 1 \end{bmatrix}$$

Compute cofactors A_{23} and A_{33}.

2. Evaluate the determinant

$$\begin{vmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & -2 & 3 \end{vmatrix}.$$

3. Let

$$A = \begin{bmatrix} 2 & -1 \\ 6 & -2 \end{bmatrix}$$

a) Write the adjoint matrix of A.
b) Find the inverse of A.

4. Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ x & y & z \\ 1 & 1 & 1 \end{bmatrix}$$

Suppose that we know

$$det(A) = 10.$$

Evaluate the following determinants:

$$\textbf{a)} \begin{vmatrix} x & y & z \\ 4 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}, \quad \textbf{b)} \begin{vmatrix} 1 & 0 & 0 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}, \quad \textbf{c)} \begin{vmatrix} 4 & 0 & 0 \\ x-2 & y-2 & z-2 \\ 1 & 1 & 1 \end{vmatrix}.$$

15.2 Answers to Homework Problems

1. $A_{23} = 1$, $A_{33} = 5$.
2. 13.
3. **a)** $adjA = \begin{bmatrix} -2 & 1 \\ -6 & 2 \end{bmatrix}$, **b)** $A^{-1} = \begin{bmatrix} -1 & \frac{1}{2} \\ -3 & 1 \end{bmatrix}$.
4. **a)** -10, **b)** $\frac{5}{2}$, **c)** 10.

16 Vectors

1. Let $P(2,-3)$, $Q(-1,2)$. Find $\vec{PQ}$ and the length of $\vec{PQ}$.

Solution.

$$\vec{PQ} = \begin{bmatrix} -1-2 \\ 2-(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$||\vec{PQ}|| = \sqrt{9+25} = \sqrt{34}$$

2. Determine the head of the vector $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ whose tail is at $P(3,2)$.

Solution. We have to determine the point $Q(x,y)$ such that $\mathbf{v} = \vec{PQ}$. We have

$$\mathbf{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \vec{PQ} = \begin{bmatrix} x-3 \\ y-2 \end{bmatrix}.$$

Therefore,

$$x - 3 = -2$$
$$x = 1,$$
$$y - 2 = 5$$
$$y = 7.$$

It follows $Q(1,7)$.

3. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

Find $2\mathbf{u} - 3\mathbf{v}$.

Solution.

$$2\mathbf{u} - 3\mathbf{v} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} + \begin{bmatrix} -15 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ 11 \end{bmatrix}$$

4. Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Find the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution.

$$\mathbf{u} \cdot \mathbf{v} = 12,$$
$$||\mathbf{u}|| = \sqrt{4+9} = \sqrt{13},$$
$$||\mathbf{v}|| = \sqrt{9+4} = \sqrt{13}.$$

Then

$$cos(\varphi) = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||} = \frac{12}{\sqrt{13}\sqrt{13}} = \frac{12}{13}.$$

5. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Find

a) $||\mathbf{u}||$, **b)** $\mathbf{u}\cdot\mathbf{v}$, **c)** $-\mathbf{u}+2\mathbf{v}$.

Solution. **a)**

$$||\mathbf{u}|| = \sqrt{1+9+4} = \sqrt{14}$$

b)

$$\mathbf{u}\cdot\mathbf{v} = 1+0+0+0+0 = 1$$

c)

$$-\mathbf{u}+2\mathbf{v} = \begin{bmatrix} -1+2 \\ 0+6 \\ -3+0 \\ 0+2 \\ 2+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -3 \\ 2 \\ 2 \end{bmatrix}$$

Example 6. Find the cosine of the angle between vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

Solution.

$$\mathbf{u}\cdot\mathbf{v} = -5,$$
$$||\mathbf{u}|| = \sqrt{25} = 5,$$
$$||\mathbf{v}|| = \sqrt{9} = 3.$$

Then

$$cos(\varphi) = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\,||\mathbf{v}||} = \frac{-5}{5\cdot 3} = -\frac{1}{3}.$$

16.1 Homework Problems

1. Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Find

a) $||\mathbf{u}||$, **b)** $\mathbf{u} \cdot \mathbf{v}$, **c)** $-2\mathbf{u} + \mathbf{v}$,
d) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

2. Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

Find

a) $||\mathbf{v}||$, **b)** $\mathbf{u} \cdot \mathbf{v}$, **c)** $3\mathbf{u} - 2\mathbf{v}$,
d) the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

16.2 Answers to Homework Problems

1. a) $\sqrt{5}$ **b)** -4 **c)** $\begin{bmatrix} 0 \\ -7 \end{bmatrix}$ **d)** $-\frac{4}{\sqrt{5} \cdot \sqrt{13}}$

2. a) 3 **b)** 3 **c)** $\begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$ **d)** $\frac{1}{\sqrt{5}}$