SOME NEW $(96, 20, 4)$ DIFFERENCE SETS

ANKA GOLEMAC AND TANJA VUČIČIĆ

ABSTRACT. The correspondence between a $(96, 20, 4)$ symmetric design having regular automorphism group and a difference set with the same parameters has been used to obtain new difference sets in some groups of order 96. New $(96, 20, 4)$ symmetric designs have been constructed under the assumption of an automorphism group $Z_4^2 \cdot Z_3$ action.

1. Introduction

A (v, k, λ) *difference set* is a k-element subset $\Delta \subseteq \Gamma$ in a group Γ of order v provided that the multiset of "differences" $\{xy^{-1} \mid x, y \in \Delta, x \neq y\}$ contains each nonidentity element of Γ exactly λ times. Δ is called *(non)abelian* difference set if Γ is (non)abelian group. By the incidence structure of its *development,* $dev\Delta = (\Gamma, \{\Delta g \mid g \in \Gamma\}, \in)$, difference set Δ is closely related to symmetric block design with the same parameters. That interrelation (details can be found in [2]) is given in the following theorem.

Theorem 1.1. *Let Γ be a finite group of order v and Δ a proper, non-empty k-element subset of Γ. Then Δ is a (v, k, λ) difference set in Γ if and only if $dev\Delta$ is a symmetric (v, k, λ) design on which Γ acts regularly.*

Let's recall, a *symmetric block design* with parameters (v, k, λ) is a finite incidence structure $\mathcal{D} = (\mathcal{V}, \mathcal{B}, \mathcal{I})$ consisting of $|\mathcal{V}| = v$ points and $|\mathcal{B}| = v$ blocks, where each block is incident with k points and any two distinct points are incident with exactly λ common blocks. An *automorphism* of a symmetric block design $\mathcal{D}$ is a permutation on $\mathcal{V}$ which sends blocks to blocks. The set of all automorphisms of $\mathcal{D}$ forms its full automorphism group denoted by $Aut\mathcal{D}$. If a subgroup $\Gamma \leq Aut\mathcal{D}$ acts regularly on $\mathcal{V}$ and $\mathcal{B}$, then $\mathcal{D}$ is called *regular* and Γ is called a *Singer group* of $\mathcal{D}$.

Two difference sets Δ^1 (in Γ^1) and Δ^2 (in Γ^2) are *isomorphic* if the designs $dev\Delta^1$ and $dev\Delta^2$ are isomorphic; Δ^1 and Δ^2 are *equivalent* if there exists a group isomorphism $\varphi : \Gamma^1 \to \Gamma^2$ such that $\varphi(\Delta^1) = \Delta^2 g$ for a suitable $g \in \Gamma^2$. It is easy to see that equivalent difference sets Δ^1 and Δ^2 give rise to isomorphic symmetric designs $dev\Delta^1$ and $dev\Delta^2$.

On the basis of Theorem 1.1 one can find difference sets in groups of order 96 through construction of $(96, 20, 4)$ symmetric designs with appropriate automorphism groups. In that sense, the action of automorphism groups $Z_2^4 \cdot Z_3$ was considered in [9]. We now complete those considerations by expanding their scope to all the groups of the type $A \cdot Z_3$, $A \ncong Z_2^4$ being an abelian group of order 16. As in [9] and for the reasons there explained, we confine ourselves to the semitransitive group action in 6 orbits of the length 16 stabilized by Z_3.

1991 *Mathematics Subject Classification.* [2000]05B05.
Key words and phrases. Symmetric design, difference set, automorphism group.

There are exactly two groups of order 48 that have the desired type: $G_1 \cong (Z_2^2 \times Z_4) \cdot Z_3$ and $G_2 \cong Z_4^2 \cdot Z_3$. In terms of generators and relations we give them in section 2 by relations (1) and (2).

2. Construction of designs

For the design construction we use the well known method of tactical decomposition, [11]. This method is based on the assumption that certain group acts on the design as its automorphism group, in which case orbit partition of points and blocks forms a tactical decomposition of the design, [3]. Tactical decompositions can be represented by orbit matrices. The entries of these matrices give the possible dispersion (regarding cardinality) of the points lying on the blocks of each block orbit into point orbits.

In particular, if the group

$$\begin{aligned} G_1 = \ & \langle a, b, c, d \mid a^4 = b^2 = c^2 = d^3 = 1, \\ & [a, b] = [a, c] = [b, c] = 1, a^d = a, b^d = c, c^d = cb \rangle \qquad \text{or} \end{aligned} \tag{1}$$

$$G_2 = \ \langle a, b, c \mid a^4 = b^4 = c^3 = 1, [a, b] = 1, a^c = a^3 b^3, b^c = a \rangle \tag{2}$$

(the notation $p^q = qpq^{-1}$, for p, q arbitrary group elements, is used) acts on a $(96, 20, 4)$ symmetric design in six orbits of the length 16, the only orbit matrix in that case is (3).

$$\begin{array}{cccccc|c} 16 & 16 & 16 & 16 & 16 & 16 & \\ \hline 0 & 4 & 4 & 4 & 4 & 4 & 16 \\ 4 & 0 & 4 & 4 & 4 & 4 & 16 \\ 4 & 4 & 0 & 4 & 4 & 4 & 16 \\ 4 & 4 & 4 & 0 & 4 & 4 & 16 \\ 4 & 4 & 4 & 4 & 0 & 4 & 16 \\ 4 & 4 & 4 & 4 & 4 & 0 & 16 \end{array} \tag{3}$$

The construction of designs corresponding to the observed tactical decomposition is equivalent to indexing orbit matrix (3). Indexing means determining precisely which points from every point orbit lie on a representative block of each block orbit. Preceded by making highly optimized programs, indexing is performed with the aid of computer. As an input to these programs we need a group G_i generators' permutation representation of degree 16, $i = 1, 2$. Those used here we give in (4) and (5).

$$G_1 \cdots \begin{cases} a = (\ 1\ 2\ 3\ 4\ )(\ 5\ 7\ 8\ 9\ )(\ 6\ 10\ 11\ 12\ )(\ 13\ 16\ 14\ 15\ ) \\ b = (\ 1\ 5\ )(\ 2\ 7\ )(\ 3\ 8\ )(\ 4\ 9\ )(\ 6\ 13\ )(\ 10\ 16\ )(\ 11\ 14\ )(\ 12\ 15\ ) \\ c = (\ 1\ 6\ )(\ 2\ 10\ )(\ 3\ 11\ )(\ 4\ 12\ )(\ 5\ 13\ )(\ 7\ 16\ )(\ 8\ 14\ )(\ 9\ 15\ ) \\ d = (\ 5\ 13\ 6\ )(\ 7\ 16\ 10\ )(\ 8\ 14\ 11\ )(\ 9\ 15\ 12\ ) \end{cases} \tag{4}$$

$$G_2 \cdots \begin{cases} a = (\ 1\ 2\ 3\ 4\ )(\ 5\ 8\ 9\ 10\ )(\ 6\ 13\ 14\ 15\ )(\ 7\ 12\ 16\ 11\ ) \\ b = (\ 1\ 5\ 6\ 7\ )(\ 2\ 8\ 13\ 12\ )(\ 3\ 9\ 14\ 16\ )(\ 4\ 10\ 15\ 11\ ) \\ c = (\ 2\ 5\ 11\ )(\ 3\ 6\ 14\ )(\ 4\ 7\ 8\ )(\ 9\ 10\ 15\ )(\ 12\ 13\ 16\ ) \end{cases} \tag{5}$$

In case of the group G_1, the completion of the indexing procedure under given assumptions proves to be impossible, i.e. the following statement holds.

Proposition 2.1. *There is no* $(96, 20, 4)$ *symmetric design on which group* G_1 *acts semitransitively in orbits of the length 16.*

For the group G_2 indexing can be completed successfully in four different ways. Precisely the following theorem holds.

Theorem 2.2. *There are exactly four nonisomorphic (96,20,4) symmetric designs with* G_2 *as an automorphism group acting on them in six orbits of the length* 16.

Proof.

As design representative blocks (six of them, each representing one block orbit) we take blocks stabilized by the subgroup $\langle c\rangle \leq G_2$. Therefore, these blocks are to be composed from $\langle c\rangle$-point orbits as a whole. If we let the numbers $1, 2, \ldots, 16$ represent the points of point orbits of our design, from (3) and (5) we easily see that there are 5^5 possibilities for a selection of 20 points of a representative block. Namely, on disposition we have 1 fixed point and 5 $\langle c\rangle$-orbits of the length three for a selection of 4 points in each point orbit. In the procedure of indexing, on each level, every possible selection of orbit representative block is submitted to all the necessary λ-balance checking, so this job is necessarily left to a computer. The procedure ends up successfully with a great number of symmetric designs constructed. For the elimination of isomorphic structures we use program by V. Krčadinac [12], which itself calls Nauty [14]. Eventually it turns out that there are exactly 4 nonisomorphic symmetric designs admitting the specified action of G_2. We give them below by their six base blocks.

D_1
$2_1\ 2_2\ 2_5\ 2_{11}\ 3_1\ 3_3\ 3_6\ 3_{14}\ 4_1\ 4_4\ 4_7\ 4_8\ 5_1\ 5_9\ 5_{10}\ 5_{15}\ 6_1\ 6_{12}\ 6_{13}\ 6_{16}$
$1_1\ 1_9\ 1_{10}\ 1_{15}\ 3_1\ 3_4\ 3_7\ 3_8\ 4_1\ 4_2\ 4_5\ 4_{11}\ 5_1\ 5_{12}\ 5_{13}\ 5_{16}\ 6_1\ 6_3\ 6_6\ 6_{14}$
$1_1\ 1_3\ 1_6\ 1_{14}\ 2_1\ 2_9\ 2_{10}\ 2_{15}\ 4_1\ 4_{12}\ 4_{13}\ 4_{16}\ 5_1\ 5_2\ 5_5\ 5_{11}\ 6_1\ 6_4\ 6_7\ 6_8$
$1_1\ 1_{12}\ 1_{13}\ 1_{16}\ 2_1\ 2_4\ 2_7\ 2_8\ 3_1\ 3_2\ 3_5\ 3_{11}\ 5_1\ 5_3\ 5_6\ 5_{14}\ 6_1\ 6_9\ 6_{10}\ 6_{15}$
$1_1\ 1_4\ 1_7\ 1_8\ 2_1\ 2_{12}\ 2_{13}\ 2_{16}\ 3_1\ 3_9\ 3_{10}\ 3_{15}\ 4_1\ 4_3\ 4_6\ 4_{14}\ 6_1\ 6_2\ 6_5\ 6_{11}$
$1_1\ 1_2\ 1_5\ 1_{11}\ 2_1\ 2_3\ 2_6\ 2_{14}\ 3_1\ 3_{12}\ 3_{13}\ 3_{16}\ 4_1\ 4_9\ 4_{10}\ 4_{15}\ 5_1\ 5_4\ 5_7\ 5_8$
D_2
$2_1\ 2_2\ 2_5\ 2_{11}\ 3_1\ 3_3\ 3_6\ 3_{14}\ 4_1\ 4_4\ 4_7\ 4_8\ 5_1\ 5_9\ 5_{10}\ 5_{15}\ 6_1\ 6_{12}\ 6_{13}\ 6_{16}$
$1_1\ 1_{12}\ 1_{13}\ 1_{16}\ 3_1\ 3_9\ 3_{10}\ 3_{15}\ 4_1\ 4_3\ 4_6\ 4_{14}\ 5_1\ 5_4\ 5_7\ 5_8\ 6_1\ 6_2\ 6_5\ 6_{11}$
$1_1\ 1_3\ 1_6\ 1_{14}\ 2_1\ 2_4\ 2_7\ 2_8\ 4_1\ 4_2\ 4_5\ 4_{11}\ 5_1\ 5_{12}\ 5_{13}\ 5_{16}\ 6_1\ 6_9\ 6_{10}\ 6_{15}$
$1_1\ 1_9\ 1_{10}\ 1_{15}\ 2_1\ 2_3\ 2_6\ 2_{14}\ 3_1\ 3_{12}\ 3_{13}\ 3_{16}\ 5_1\ 5_2\ 5_5\ 5_{11}\ 6_1\ 6_4\ 6_7\ 6_8$
$1_1\ 1_4\ 1_7\ 1_8\ 2_1\ 2_{12}\ 2_{13}\ 2_{16}\ 3_1\ 3_2\ 3_5\ 3_{11}\ 4_1\ 4_9\ 4_{10}\ 4_{15}\ 6_1\ 6_3\ 6_6\ 6_{14}$
$1_1\ 1_2\ 1_5\ 1_{11}\ 2_1\ 2_9\ 2_{10}\ 2_{15}\ 3_1\ 3_4\ 3_7\ 3_8\ 4_1\ 4_{12}\ 4_{13}\ 4_{16}\ 5_1\ 5_3\ 5_6\ 5_{14}$
D_3
$2_1\ 2_2\ 2_5\ 2_{11}\ 3_1\ 3_3\ 3_6\ 3_{14}\ 4_1\ 4_4\ 4_7\ 4_8\ 5_1\ 5_9\ 5_{10}\ 5_{15}\ 6_1\ 6_{12}\ 6_{13}\ 6_{16}$
$1_1\ 1_{12}\ 1_{13}\ 1_{16}\ 3_1\ 3_9\ 3_{10}\ 3_{15}\ 4_1\ 4_3\ 4_6\ 4_{14}\ 5_1\ 5_2\ 5_5\ 5_{11}\ 6_1\ 6_4\ 6_7\ 6_8$
$1_1\ 1_3\ 1_6\ 1_{14}\ 2_1\ 2_4\ 2_7\ 2_8\ 4_1\ 4_2\ 4_5\ 4_{11}\ 5_1\ 5_{12}\ 5_{13}\ 5_{16}\ 6_1\ 6_9\ 6_{10}\ 6_{15}$
$1_1\ 1_9\ 1_{10}\ 1_{15}\ 2_1\ 2_3\ 2_6\ 2_{14}\ 3_1\ 3_{12}\ 3_{13}\ 3_{16}\ 5_1\ 5_4\ 5_7\ 5_8\ 6_1\ 6_2\ 6_5\ 6_{11}$
$1_1\ 1_4\ 1_7\ 1_8\ 2_1\ 2_{12}\ 2_{13}\ 2_{16}\ 3_1\ 3_2\ 3_5\ 3_{11}\ 4_1\ 4_9\ 4_{10}\ 4_{15}\ 6_1\ 6_3\ 6_6\ 6_{14}$
$1_1\ 1_2\ 1_5\ 1_{11}\ 2_1\ 2_9\ 2_{10}\ 2_{15}\ 3_1\ 3_4\ 3_7\ 3_8\ 4_1\ 4_{12}\ 4_{13}\ 4_{16}\ 5_1\ 5_3\ 5_6\ 5_{14}$
D_4
$2_1\ 2_3\ 2_6\ 2_{14}\ 3_1\ 3_4\ 3_7\ 3_8\ 4_1\ 4_9\ 4_{10}\ 4_{15}\ 5_1\ 5_{12}\ 5_{13}\ 5_{16}\ 6_1\ 6_2\ 6_5\ 6_{11}$
$1_1\ 1_3\ 1_6\ 1_{14}\ 3_1\ 3_9\ 3_{10}\ 3_{15}\ 4_1\ 4_4\ 4_7\ 4_8\ 5_1\ 5_2\ 5_5\ 5_{11}\ 6_1\ 6_{12}\ 6_{13}\ 6_{16}$
$1_1\ 1_4\ 1_7\ 1_8\ 2_1\ 2_{12}\ 2_{13}\ 2_{16}\ 4_1\ 4_2\ 4_5\ 4_{11}\ 5_1\ 5_3\ 5_6\ 5_{14}\ 6_1\ 6_9\ 6_{10}\ 6_{15}$

1_1 1_{12} 1_{13} 1_{16} 2_1 2_4 2_7 2_8 3_1 3_2 3_5 3_{11} 5_1 5_9 5_{10} 5_{15} 6_1 6_3 6_6 6_{14}
1_1 1_9 1_{10} 1_{15} 2_1 2_2 2_5 2_{11} 3_1 3_3 3_6 3_{14} 4_1 4_{12} 4_{13} 4_{16} 6_1 6_4 6_7 6_8
1_1 1_2 1_5 1_{11} 2_1 2_9 2_{10} 2_{15} 3_1 3_{12} 3_{13} 3_{16} 4_1 4_3 4_6 4_{14} 5_1 5_4 5_7 5_8
The points of the design are denoted by $\mathrm{I}_1,\mathrm{I}_2,\ldots,\mathrm{I}_{16}$, I = 1,2,. . .,6 as accustomed. The subgroup $\langle a, b\rangle \leq G_2$ generates all the blocks of the designs. ∎

Taking an incidence matrix of the design D_i ($i = 1,\ldots,4$) as an input, the computer program by V. Tonchev [15] gives out the order of the full automorphism group $AutD_i$, as well as a permutation representation of degree 96 of $AutD_i$ generators. The latter enables our further analysis of the properties of $AutD_i$ with the help of GAP, system for computational group theory, [7]. In order to be systematic (since we deal with a large number of groups), when denoting groups henceforth we indicate their GAP Library Small Groups catalogue number, 'GAP-cn' for short. It is of the form $[m, j]$ which stands for j-th group of order m in the catalogue, $m \leq 2000$. In that sense G_1 and G_2 are referred to as $[48, 31]$ and $[48, 3]$ respectively.

About the groups $AutD_i, i = 1,\ldots,4$, we found the following.

Theorem 2.3. *The groups $AutD_3$ and $AutD_4$ act transitively on the designs D_3 and D_4 respectively; $| AutD_3 |=| AutD_4 |= 576$ and $AutD_3 \ncong AutD_4$. The groups $AutD_1$ and $AutD_2$ act on designs D_1 and D_2 in two orbits of the lengths 32 and 64. $| AutD_1 |=| AutD_2 |= 192$ and $AutD_1 \ncong AutD_2$.*

Remark 2.1. *In comparing these results to our previous ones that refer to parameters* $(96, 20, 4)$, *we used* [12] *to check that none $D_i, i = 1,\ldots,4$ is isomorphic to any design given in* [9]*, and that D_4 is isomorphic to the only design in* [8]*. As for the symmetric designs with these parameters explicitly presented in literature, it's easy to see that here constructed designs $D_i, i = 1,\ldots,4$ are not isomorphic to the three designs cited in* [4]*. The assertion follows from the fact that the orders of their full automorphism groups differ. Further, in* [5]*, p.302, one finds three mutually nonisomorphic designs in the form of abelian difference sets, while the design given on p. 82 is isomorphic to the development of the third difference set mentioned. By using* [12]*, the designs $D_i, i = 1,\ldots,4$ are checked not to be isomorphic to any of these designs.*

3. Construction of difference sets

Starting from the detected transitive action of $AutD_3$ and $AutD_4$ (Theorem 2.3), in intended construction of $(96, 20, 4)$ difference sets we proceed with the inspection of Singer groups of D_3 and D_4 in GAP. More details on the procedure can be found in [9]. As we have indicated in Remark 2.1, the procedure for $AutD_4$ (GAP-cn [576, 5566]) has already been made in [8]. There we found that $AutD_4$ has five subgroups, all nonabelian, that act regularly on D_4 and consequently the existence status regarding difference sets was solved for the groups with the following GAP-cns: [96, 72], [96, 78], [96, 147], [96, 174], and [96, 209]. Thus, the inspection of $AutD_3$ (GAP-cn [576, 5550]) for its regular subgroups will conclude our investigation of $(96, 20, 4)$ difference sets under given terms.

In terms of generators and relations we have:

$$\begin{aligned} AutD_3 = \quad & \langle a, b, c, d, e, f \mid a^4 = b^4 = [a, b] = c^2 = [a, c] = [b, c] = 1, d^3 = 1, \\ & a^d = a^3b^3, b^d = a, [c, d] = 1, e^3 = 1, a^e = a^3b^3, b^e = a, [c, e] = 1, \\ & [d, e] = f^2 = 1, a^f = a^3b^2, b^f = a^2b, [c, f] = 1, d^f = e^2d^2, [e, f] = 1\rangle. \end{aligned}$$

A permutation representation of the $AutD_3$ generators, obtained by using [15], is given in Table 1.

TABLE 1. *Generators of* $AutD_3$,*GAP-cn :[576,5550]*

r1:=(2,13)(4,15)(5,16)(7,9)(8,10)(11,12)(17,81)(18,93)(19,83)(20,95)(21,96)(22,86)(23,89)(24,90)(25,87)(26,88)(27,92)(28,91)(29,82)(30,94)(31,84)(32,85)(34,45)(36,47)(37,48)(39,41)(40,42)(43,44)(49,65)(50,77)(51,67)(52,79)(53,80)(54,70)(55,73)(56,74)(57,71)(58,72)(59,76)(60,75)(61,66)(62,78)(63,68)(64,69)

r2:=(2,5,11)(3,6,14)(4,7,8)(9,10,15)(12,13,16)(18,21,27)(19,22,30)(20,23,24)(25,26,31)(28,29,32)(34,37,43)(35,38,46)(36,39,40)(41,42,47)(44,45,48)(50,53,59)(51,54,62)(52,55,56)(57,58,63)(60,61,64)(66,69,75)(67,70,78)(68,71,72)(73,74,79)(76,77,80)(82,85,91)(83,86,94)(84,87,88)(89,90,95)(92,93,96)

r3:=(1,2,3,4)(5,8,9,10)(6,13,14,15)(7,12,16,11)(17,18,19,20)(21,24,25,26)(22,29,30,31)(23,28,32,27)(33,34,35,36)(37,40,41,42)(38,45,46,47)(39,44,48,43)(49,50,51,52)(53,56,57,58)(54,61,62,63)(55,60,64,59)(65,66,67,68)(69,72,73,74)(70,77,78,79)(71,76,80,75)(81,82,83,84)(85,88,89,90)(86,93,94,95)(87,92,96,91)

r4:=(1,17)(2,29)(3,19)(4,31)(5,32)(6,22)(7,25)(8,26)(9,23)(10,24)(11,28)(12,27)(13,18)(14,30)(15,20)(16,21)(33,49)(34,61)(35,51)(36,63)(37,64)(38,54)(39,57)(40,58)(41,55)(42,56)(43,60)(44,59)(45,50)(46,62)(47,52)(48,53)(65,81)(66,93)(67,83)(68,95)(69,96)(70,86)(71,89)(72,90)(73,87)(74,88)(75,92)(76,91)(77,82)(78,94)(79,84)(80,85)

Computation in GAP, with the content of Table 1 as an input, reveals the total of four Singer subgroups in $AutD_3$. We give them in terms of generators and relations, following the labelling used in [9]. The indices suggest GAP-cns.

$$\begin{aligned} H_{[96,68]} = \quad & \big\langle x, y, z, w \mid x^4 = y^4 = [x, y] = 1, z^2 = [x, z] = [y, z] = 1, \\ & \qquad w^3 = 1, x^w = x^3y^3, y^w = x, [z, w] = 1 \big\rangle, \\ H_{[96,83]} = \quad & \big\langle x, y, z, w \mid x^4 = y^4 = [x, y] = 1, z^2 = 1, x^z = x^3y^2, y^z = x^2y, \\ & \qquad w^3 = [x, w] = [y, w] = 1, w^z = w^2 \big\rangle, \\ H_{[96,130]} = \quad & \big\langle x, y, z, w \mid x^4 = y^4 = 1, y^x = y^3, z^2 = 1, [x, z] = [y, z] = 1, \\ & \qquad w^3 = [x, w] = 1, w^y = w^2, [z, w] = 1 \big\rangle, \\ H_{[96,161]} = \quad & Z_4^2 \times Z_2 \times Z_3 = \langle x, y\rangle \times \langle z\rangle \times \langle w\rangle. \end{aligned}$$

Because $AutD_3$ acts transitively on it, D_3 can be represented by a single block, for instance by

D_3 := [1, 12, 13, 16, 33, 41, 42, 47, 49, 51, 54, 62, 65, 66, 69, 75, 81, 84, 87, 88]. Now {1,2,...,96} is taken as the set of points of our design. By identifying the points of D_3 with the elements of its Singer groups, in each of them we obtain the corresponding difference set. For more details on this identification see [10] or [16]. In $H_{[96,83]}$, up to equivalence, we find two difference sets, because $AutD_3$ has two conjugacy classes of subgroups isomorphic to $H_{[96,83]}$. In a customary manner, difference sets are presented as elements of the integral group ring $\mathbb{Z}H_{[96,n]}$, where $H_{[96,n]} = \left\{w^lx^py^jz^k \mid l = 0, 1, 2; p, j = 0, \ldots, 3; k = 0, 1\right\}, n = 68, 83, 130$, and 161.

$$\begin{aligned}
\Delta_{[96,68]} = {} & 1 + x + y + x^3y^3 + z + xyz + x^3z + y^3z \\
& + w(x + y + y^2 + x^3y + xz + x^2y^2z + x^2y^3z + x^3y^3z) \\
& + w^2(xy + xy^3 + x^3y + x^3y^3); \\
\Delta^1_{[96,83]} = {} & 1 + x + xy^3 + x^2y + xyz + x^3z + x^2yz + x^2y^2z \\
& + w(1 + x^2 + y^2 + x^2y^2 + yz + xy^2z + xy^3z + x^2y^2z) \\
& + w^2(1 + y + xy + x^3y^2); \\
\Delta^2_{[96,83]} = {} & 1 + x + y + x^3y + x^2z + y^3z + x^3yz + x^3y^2z \\
& + w(1 + x^2y + x^3y^2 + x^3y^3) \\
& + w^2(1 + x^2 + y^2 + x^2y^2 + xz + x^2z + x^2y^3z + x^3y^3z); \\
\Delta_{[96,130]} = {} & 1 + x + y^3 + xy^3 + y^2z + xy^3z + x^2y^3z + x^3y^2z \\
& + w(1 + xy^2 + x^2y^3 + x^3y + x^3z + y^2z + y^3z + x^3yz) \\
& + w^2(1 + x^2 + y^2 + x^2y^2); \\
\Delta_{[96,161]} = {} & 1 + x + xy + x^2y^3 + z + x^3z + x^2yz + x^3y^3z \\
& + w(1 + x^2 + y^2 + x^2y^2) \\
& + w^2(1 + y^3 + xy^3 + x^3y^2 + z + yz + xy^2z + x^3yz).
\end{aligned}$$

The relations $dev\Delta_{[96,n]} \cong D_3$ hold for all difference sets obtained.

Remark 3.1. *Group $H_{[96,161]}$ is abelian. The existence of difference sets in it has been proved in* [1]. *As for the remaining three groups: $H_{[96,68]}, H_{[96,83]}$, and $H_{[96,130]}$, here given constructions of $\Delta_{[96,68]}, \Delta^i_{[96,83]}$, $i = 1, 2$, and $\Delta_{[96,130]}$ are a proof of existence of difference sets in them.*

$(96, 20, 4)$ difference sets belong to the McFarland series with parameters of the form $\left(q^{d+1}(1 + \frac{q^{d+1}-1}{q-1}), q^d \frac{q^{d+1}-1}{q-1}, q^d \frac{q^d-1}{q-1}\right)$, where q is any prime power and d is any positive integer. It is worth pointing out that here given difference sets cannot be obtained using the well known construction by McFarland and Dillon ([13], [6]), as the center of none of the groups $H_{[96,n]}$, $n = 68, 83, 130$, and 161, contains elementary abelian subgroup E_{16} of order $q^{d+1} = 16$. Namely, $Z(H_{[96,68]}) \cong Z_2, Z(H_{[96,83]}) \cong Z_2^2$, and $Z(H_{[96,130]}) \cong Z_2^3$.

4. Literature sources

Let's focus a bit more on three difference sets, i. e. symmetric designs given explicitly in [5], page 302.

The first one, in the group $Z_2 \times Z_4^2 \times Z_3$ (GAP-cn: [96, 161]) is given incorrectly. For the correct citing see [1]. This difference set is not isomorphic (and consequently not equivalent) to $\Delta_{[96,161]}$. We found that the full automorphism group of its development is the group [192,1038] of order 192. It has eight subgroups that act regularly on this design. Besides $H_{[96,161]}$, them being nonabelian groups with the following GAP-cns: [96, 75], [96, 77], [96, 79], [96, 83], [96, 131], [96, 135], and [96, 136]. We easily checked that the difference set in [96, 83] is not isomorphic to $\Delta^1_{[96,83]}$ and $\Delta^2_{[96,83]}$ given in Section 3. For the remaining six groups this solves the existence status regarding difference sets.

The second, in the group $Z_2^3 \times Z_4 \times Z_3$ (GAP-cn: [96, 220]), has the full automorphism group of order 96, hence $Z_2^3 \times Z_4 \times Z_3$ is the only Singer group of the design.

The third, a difference set in the group $Z_2^5 \times Z_3$ ([96, 231]), has the full automorphism group of order 576, more precisely [576, 5603]. Its development is isomorphic to the symmetric design given in [9] and there denoted by D_2. It was shown in [9] that, besides $H_{[96,231]}$, Singer groups of D_2 were the following three groups: [96, 159], [96, 160], and [96, 229]. The difference sets in these groups were given explicitly.

Altogether, on grounds of the preceding review and Section 3, the list of 35 groups of order 96 given in [9], for which the question of existence of difference sets is solved, can be expanded with 6+3=9 new groups. Added the five groups given in [8], we come to the list of 49 groups.

References

[1] K.T. Arasu and S.K. Sehgal, Some new difference sets, J. Combin. Theory Ser. A, 69 (1995) pp. 170-172.

[2] T. Beth, D. Jungnickel, and H. Lenz, *Design theory*, Cambridge University Press (1999).

[3] A. Beutelspacher, *Einführung in die endliche Geometrie I*, Bibliographisches Institut, Mannheim-Wien-Zürich (1985).

[4] A.E. Brouwer, J.H. Koolen & M.H. Klin, A root graph that is locally the line graph of the Petersen graph, Discrete Mathematics, 264 (2003) pp. 13-24.

[5] C.J. Colbourn and J.H. Dinitz, Eds., *The CRC Handbook of Combinatorial Designs*, CRC Press, New York (1996); URL of the home page with new results is http://www.emba.uvm.edu/~dinitz/newresults

[6] J.F. Dillon, Variations on a scheme of McFarland for noncyclic difference sets, J. Combin. Theory Ser. A, 40 (1985) pp. 9-21.

[7] [GAP 99] The GAP Group, GAP - - - Groups, Algorithms, and Programming, Version 4.1; Aachen, St Andrews (1999).

[8] A. Golemac, J. Mandić and T. Vučičić, One (96,20,4) Symmetric Design and Related Nonabelian Difference Sets, to appear in Designs, Codes and Cryptography; http://www.pmfst.hr/~vucicic/DS(96,20,4)n1.pdf.

[9] A. Golemac, J. Mandić and T. Vučičić, On the Existence of Difference Sets in Groups of Order 96, submitted; http://www.pmfst.hr/~vucicic/DS(96,20,4)n2.pdf.

[10] A. Golemac and T. Vučičić, New difference sets in nonabelian groups of order 100, Journal of Combinatorial Designs, 9 (2001) pp. 424-434.

[11] Z. Janko, Coset enumeration in groups and constructions of symmetric designs, Combinatorics 90, Elsevier Science Publishers, (1992) pp. 275-277.

[12] V. Krčadinac, Steiner 2-designs $S(k, 2k^2 - 2k + 1)$, MSc Thesis, University of Zagreb (1999).

[13] R.L. McFarland, A family of difference sets in non-cyclic groups, J. Combin. Theory Ser. A, 15 (1973) pp. 1-10.

[14] B.D. McKay, Nauty user's guide (version 1.5), Technical Report TR-CS-90-02, Computer Science Department, Australian National University (1990).

[15] V. Tonchev, Michigan Technological University, private communication via Z. Janko of Universität Heidelberg.

[16] T. Vučičić, New symmetric designs and nonabelian difference sets with parameters (100,45,20), Journal of Combinatorial Designs, 8 (2000) pp. 291-299.

University of Split, Dept. of mathematics, Teslina 12/III, 21 000 Split, Croatia
e-mail: golemac@mapmf.pmfst.hr
e-mail: vucicic@mapmf.pmfst.hr