Theorem 1.1 $\alpha\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8,m_{11},m_{12},m_{13}, \\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44} \end{pmatrix}=1$ if and only of one of the following statements holds:

1) $(i_7=0)$ and $\beta\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_8,m_{11},m_{12},m_{13}, \\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},0 \end{pmatrix}=1$.

2) $(i_7\neq 0)$ and $(i_4=0)$ and

$$\left(\exists i\in\{0,1,...,\min\{m_{12},m_{13},m_{23}\}\}\right)\beta\begin{pmatrix} i_1,i_2,i_3,2,i_5,i_6,i_8,m_{11},m_{12}-i,m_{13}+i, \\ m_{14},m_{22},m_{23}-i,m_{24},m_{33},m_{34},m_{44},i \end{pmatrix}=1$$

3) $(i_7\neq 0)$ and $(i_4\neq 0)$ and

$$\left(\exists i\in\{0,1,...,\min\{m_{12},m_{13},m_{23}\}\}\right)\beta\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_8,m_{11},m_{12}-i,m_{13}+i, \\ m_{14},m_{22},m_{23}-i,m_{24},m_{33},m_{34},m_{44},i \end{pmatrix}=1$$

Proof: First, suppose that $\alpha\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8,m_{11},m_{12},m_{13}, \\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44} \end{pmatrix}=1$. Let G be any graph from the set $A\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8,m_{11},m_{12},m_{13}, \\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44} \end{pmatrix}$. Distinguish three cases:

CASE 1: $i_7=0$.

Note that $G\in B\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_8,m_{11},m_{12},m_{13}, \\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}$. Therefore,

$$\beta\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_8,m_{11},m_{12},m_{13}, \\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},0 \end{pmatrix}=1.$$

CASE 2: $(i_7\neq 0)$ and $(i_4=0)$.

Let G' be a graph obtained by replacing each (2-methyl-propyl)-thorn by isopropyl thorn. Denote by i the number of (2-methyl-propyl)-thorns in G. Obviously $i\in\{0,1,...,\min\{m_{12},m_{13},m_{23}\}\}$; $\mu(G')=\begin{pmatrix} m_{11},m_{12}-i,m_{13}+i,m_{14},m_{22}, \\ m_{23}-i,m_{24},m_{33},m_{34},m_{44} \end{pmatrix}$ and G' has exactly i isopropyl thorns. It follows that $G'\in B\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_8,m_{11},m_{12}-i,m_{13}+i, \\ m_{14},m_{22},m_{23}-i,m_{24},m_{33},m_{34},m_{44},i \end{pmatrix}$. Hence, indeed

$$\left(\exists i\in\{0,1,...,\min\{m_{12},m_{13},m_{23}\}\}\right)\beta\begin{pmatrix} i_1,i_2,i_3,2,i_5,i_6,i_8,m_{11},m_{12}-i,m_{13}+i, \\ m_{14},m_{22},m_{23}-i,m_{24},m_{33},m_{34},m_{44},i \end{pmatrix}=1.$$

CASE 3: $(i_7\neq 0)$ and $(i_4\neq 0)$.

Let G' be a graph obtained by replacing each (2-methyl-propyl)-thorn by isopropyl thorn. Denote by i the number of (2-methyl-propyl)-thorns in G. Obviously $i\in\{0,1,...,\min\{m_{12},m_{13},m_{23}\}\}$; $\mu(G')=\begin{pmatrix} m_{11},m_{12}-i,m_{13}+i,m_{14},m_{22}, \\ m_{23}-i,m_{24},m_{33},m_{34},m_{44} \end{pmatrix}$ and G' has at least i

isopropyl thorns. It follows that $G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12} - i, m_{13} + i, \\ m_{14}, m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, i \end{pmatrix}$. Hence, indeed

$\left(\exists i \in \{0,1,...,\min\{m_{12}, m_{13}, m_{23}\}\}\right)\beta\begin{pmatrix} i_1, i_2, i_3, 2, i_5, i_6, i_8, m_{11}, m_{12} - i, m_{13} + i, \\ m_{14}, m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, i \end{pmatrix} = 1$.

Now, let us prove the opposite implication. First suppose that 1) holds. Let

$G \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$ be an arbitrary graph. Note that

$G \in A\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44} \end{pmatrix}$. Therefore,

$\alpha\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44} \end{pmatrix} = 1$.

Now, suppose that 2) holds. Let $G \in B\begin{pmatrix} i_1, i_2, i_3, 2, i_5, i_6, i_8, m_{11}, m_{12} - i, m_{13} + i, \\ m_{14}, m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, i \end{pmatrix}$ be an arbitrary graph. Note that in G there are exactly i isopropyl thorns. Let G' be a graph obtained by replacing each of these thorns by a (2-methyl-propyl)-thorn. Note that G' doesn't have any isopropyl thorns, hence indeed

$G \in A\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, m_{11}, m_{12} - i + i, m_{13} + i - i, \\ m_{14}, m_{22}, m_{23} - i + i, m_{24}, m_{33}, m_{34}, m_{44} \end{pmatrix} = A\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44} \end{pmatrix}$.

If 3) holds, the proof is completely analogous as in the previous case. ■

Let function $\gamma : \{0,1\}^2 \times \{0,1,2\} \times \{0,1\}^2 \times N_0^{11} \to \{0,1\}$ be given by

$\gamma\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$ if and only "There is a thorny cycle G such that

$\mu(G) = \begin{pmatrix} m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44} \end{pmatrix}$ with the following thorn allowed: methyl-thorn (if $i_1 = 1$); ethyl-thorn (if $j_2 = 1$); isopropyl-thorn (if $i_4 > 1$); (2-methyl-propyl)-thorn (if $i_6 = 1$); (2-methyl-propyl)-thorn (if $i_7 = 1$); and (1,1-dimethyl-ethyl)-thorn (if $i_8 = 1$) and there are at least r isopropyl-thorns (if $i_4 \neq 2$) or there are exactly r isopropyl-thorns (if $i_4 = 2$). Denote the set of all graphs by with these required properties by $C\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$.

Lemma 1.2

$$\left(\exists(m_{22} \in N)\right)\left(\gamma\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1\right) \Rightarrow$$

$$\Rightarrow \left(\forall(m_{22}' \in N, m_{22}' \geq m_{22})\right)\left(\gamma\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}', m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1\right)$$

Proof: Let $G \in C\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that no edge that connects two vertices of degree 2 can not be a part of any thorn. Therefore, it is a part of a cycle. Let G' be a graph obtained by replacing an arbitrary of these edges by a path of length $m_{22}'-(m_{22}-1)$.

Note that $G'\in C\begin{pmatrix} j_2,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22}',m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}$, hence

$\gamma\begin{pmatrix} j_2,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22}',m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1$.□

Theorem 1.3 $\beta\begin{pmatrix} i_1,i_2,i_3,i_4,i_5,i_6,i_8,m_{11},m_{12},m_{13}, \\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1$ if and only if one of the following statements hold:

1) $(i_3=0)$ and $(i_5=0)$ and $\left(\gamma\begin{pmatrix} i_2,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

2) $(i_2=0)$ and $(i_3=1)$ and $(i_5=0)$ and $(m_{22}\geq m_{12})$ and $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22}-m_{12},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

3) $(i_2=1)$ and $(i_3=1)$ and $(i_5=0)$ and one of the following holds:

3.1) $(m_{12}\geq m_{22})$ and $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

3.2) $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

4) $(i_2=0)$ and $(i_3=0)$ and $(i_5=1)$ and $(m_{22}\geq 2\cdot m_{12})$ and $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22}-2\cdot m_{12},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

5) $(i_2=1)$ and $(i_3=0)$ and $(i_5=1)$ and one of the following holds:

5.1) $(2\cdot m_{12}\geq m_{22})$ and $(m_{22}=0(\mathrm{mod}\,2))$ and $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

5.2) $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22}-m_{12},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

6) $(i_2=0)$ and $(i_3=1)$ and $(i_5=1)$ and $(m_{22}\geq m_{12})$ and one of the following holds:

6.1) $(2\cdot m_{12}\geq m_{22})$ and $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

6.2) $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ m_{22}-m_{12},m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

7) $(i_2=1)$ and $(i_3=1)$ and $(i_5=1)$ and one of the following holds:

7.1) $(2\cdot m_{12}\geq m_{22})$ and $\left(\gamma\begin{pmatrix} 1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14}, \\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r \end{pmatrix}=1\right)$

7.2) $\left(\gamma\begin{pmatrix}1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}=1\right)$

Proof: First, let us suppose that $\beta\begin{pmatrix}i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}=1$ and let $G \in B\begin{pmatrix}i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}$. Distinguish seven cases:

CASE 1: $(i_3 = 0)$ and $(i_5 = 0)$.

Note that $G \in C\begin{pmatrix}i_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}$, hence indeed $\gamma\begin{pmatrix}i_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}=1$.

CASE 2: $(i_2 = 0)$ and $(i_3 = 1)$ and $(i_5 = 0)$.

Note that edges of type e_{12} in G are part of prophyl-thorns. Therefore, indeed $(m_{22} \geq m_{12})$. Let G' be a graph obtained from graph G by replacing each of the propyl-thorns by ethyl-thorns. Note that $G' \in C\begin{pmatrix}1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}-m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}$. Hence, $\gamma\begin{pmatrix}1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}-m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}=1$.

CASE 3: Denote by t number of propyl-thorns in G. Note that $m_{12} \geq t$. Let G' be a graph obtained from graph G by replacing each of the propyl-thorns by ethyl-thorns. Note that $G' \in C\begin{pmatrix}1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}-t, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}$. Therefore, $\gamma\begin{pmatrix}1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}-t, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}=1$.

Distinguish two subcases:

SUBCASE 3.1: $m_{22}-t=0$.

Note that $m_{12} \geq t = m_{22}$, hence 3.1) holds.

SUBCASE 3.2: $m_{22}-t>0$.

Note that $m_{22} \geq m_{22}-t$, hence the claim follows from the Lemma 1.2

CASE 4: $(i_2 = 0)$ and $(i_3 = 1)$ and $(i_5 = 0)$.

Note that edges of type e_{12} in G are part of buthyl-thorns. Therefore, indeed $(m_{22} \geq 2 \cdot m_{12})$. Let G' be a graph obtained from graph G by replacing each of the buthyl-thorns by ethyl-thorns. Note that $G' \in C\begin{pmatrix}1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}-2\cdot m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}$. Hence, $\gamma\begin{pmatrix}1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}-2\cdot m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r\end{pmatrix}=1$.

CASE 5: Denote by t number of buthyl-thorns in G. Note that $m_{12} \geq t$. Let G' be a graph obtained from graph G by replacing each of the propyl-thorns by ethyl-thorns. Note that

$G' \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - 2 \cdot t, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therfore,

$\gamma\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - 2 \cdot t, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Distinguish two subcases:

SUBCASE 5.1: $m_{22} - 2 \cdot t = 0$.

Note that $m_{12} \geq t = \frac{m_{22}}{2}$ and that m_{22} is even, hence 5.1) holds.

SUBCASE 5.2: $m_{22} - 2 \cdot t > 0$.

Note that $m_{22} \geq m_{22} - 2 \cdot t$, hence the claim follows from the Lemma 1.2

CASE 6: Denote by t_p number of propyl-thorns in G and by t_b number of buthyl thorn. Note that $m_{12} = t_p + t_b \leq m_{22}$. Let G' be a graph obtained from graph G by replacing each of the propyl-thorns by ethyl-thorns. Note that $G' \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - t_p - 2t_b, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$.

Therefore, $\gamma\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - t_p - 2t_b, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Distinguish two subcases:

SUBCASE 6.1: $m_{22} - t_p - 2t_b = 0$.

Note that $m_{22} = t_p + 2t_b \leq 2(t_p + t_b) = 2m_{12}$ is even, hence 6.1) holds.

SUBCASE 6.2: $m_{22} - t_p - 2t_b > 0$.

Note that $m_{22} \geq m_{22} - t_p - 2t_b$, hence the claim follows from the Lemma 1.2

CASE 7: Denote by t_p number of propyl-thorns in G and by t_b number of buthyl thorn. Let G' be a graph obtained from graph G by replacing each of the propyl-thorns by ethyl-thorns. Note that $G' \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - t_p - 2t_b, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therefore,

$\gamma\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - t_p - 2t_b, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Distinguish two subcases:

SUBCASE 7.1: $m_{22} - t_p - 2t_b = 0$.

Note that $m_{22} = t_p + 2t_b \leq 2(t_p + t_b) \leq 2m_{12}$ is even, hence 7.1) holds.

SUBCASE 7.2: $m_{22} - t_p - 2t_b > 0$.

Note that $m_{22} \geq m_{22} - t_p - 2t_b$, hence the claim follows from the Lemma 1.2

Now, let us prove the opposite implication.

First, suppose that 1) holds. Let $G \in C\begin{pmatrix} i_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that $G \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therefore, $\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Now, suppose that 2) holds. Let $G \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$ Note that all edges of type e_{12} are part of ethyl-thorn. Let G' be a graph obtained from graph G by replacing each ethyl-thorn by propyl-thorns. Note that $G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therofore, $\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Suppose that $(i_2 = 1)$ and $(i_3 = 1)$ and $(i_5 = 0)$ and that 3.1) holds. Let $G \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ 0, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that G has exactly m_{12} ethyl-thorns. Pick arbitary $m_{22}\ (\le m_{12})$ ethyl-thorns. Let G' be a graph obtained by replacing these thorn by propyl-thorns. Note that $G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therofore, $\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Suppose that $(i_2 = 1)$ and $(i_3 = 1)$ and $(i_5 = 0)$ and that 3.2) holds. Let $G \in C\begin{pmatrix} i_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that $G \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therefore, $\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Now, suppose that 4) holds. Let $G \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - 2 \cdot m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$ Note that all edges of type e_{12} are part of ethyl-thorn. Let G' be a graph obtained from graph G by replacing each ethyl-thorn by buthyl-thorns. Note that $G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therofore, $\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Suppose that $(i_2 = 1)$ and $(i_3 = 0)$ and $(i_5 = 1)$ and that 5.1) holds. Let $G \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ 0, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that G has exactly m_{12} ethyl-thorns. Pick arbitary $\frac{m_{22}}{2}(\le m_{12})$ ethyl-thorns. Let G' be a graph obtained by replacing these thorn by buthyl-thorns. Note that $G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therefore,

$$\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1.$$

Suppose that $(i_2 = 1)$ and $(i_3 = 0)$ and $(i_5 = 1)$ and that 5.2) holds. Let $G \in C\begin{pmatrix} i_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that $G \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therefore, $\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Suppose that $(i_2 = 0)$ and $(i_3 = 1)$ and $(i_5 = 1)$ and $(m_{22} \ge m_{12})$ and 6.1) holds. Since $m_{12} \le m_{22} \le 2m_{12}$, there are numbers $t_p, t_b \in N_0$ such that $t_p + 2t_b = m_{22}$ and that $t_p + t_b = m_{12}$. Let $G \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ 0, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that G has exactly m_{12} ethyl-thorns. Let G' be a graph obtained from G by replacing t_p of this thorns by propyl-thorns and remaining t_b thorns by buthyl-thorns. Note that $G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therefore,

$$\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1.$$

Suppose that $(i_2 = 0)$ and $(i_3 = 1)$ and $(i_5 = 1)$ and $(m_{22} \ge m_{12})$ and 6.2) holds. Let $G \in \begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Let $G \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22} - m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$

Note that all edges of type e_{12} are part of ethyl-thorn. Let G' be a graph obtained from graph G by replacing each ethyl-thorn by propyl-thorns. Note that

$G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therofore,

$$\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1.$$

Suppose that $(i_2 = 1)$ and $(i_3 = 1)$ and $(i_5 = 1)$ and 7.1) holds. Since $m_{22} \le 2m_{12}$, there are numbers $t_e, t_p, t_b \in N_0$ such that $t_p + 2t_b = m_{22}$ and that $t_e + t_p + t_b = m_{12}$. Let $G \in C\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ 0, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that G has exactly m_{12} ethyl-thorns. Let G' be a graph obtained from G by replacing t_p of this thorns by propyl-thorns and t_b thorns by buthyl-thorns. Note that $G' \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Therefore,

$\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

Suppose that $(i_2 = 1)$ and $(i_3 = 1)$ and $(i_5 = 1)$ and 7.2) holds. Let Let

$G \in C\begin{pmatrix} i_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$. Note that $G \in B\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$.

Therefore, $\beta\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, \\ m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix} = 1$.

All the cases are exhausted and the theorem is proved. □

Let function $\delta : \{0,1\}^2 \times \{0,1,2\} \times \{0,1\} \times N_0^{12} \to \{0,1\}$ be given by

$\delta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if "There is a thorny cycle G such that

$\mu(G) = \begin{pmatrix} m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44} \end{pmatrix}$ with the following thorn allowed: methyl-thorn (if $j_1 = 1$); ethyl-thorn (if $j_2 = 1$); isopropyl-thorn (if $j_3 > 1$); and (1,1-dimethyl-ethyl)-thorn (if $j_4 = 1$) there are at least r isopropyl-thorns (if $i_4 \neq 2$) or there are exactly r isopropyl-thorns (if $i_4 = 2$); and there are at least s ethyl-thorns . Denote the set of all graphs by with these required properties by $D\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix}$.

Theorem 1.4 $\gamma\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if one of the following holds:

1) $(i_6 = 0)$ and $\delta\begin{pmatrix} i_1, j_2, i_4, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, 0 \end{pmatrix} = 1$

2) $(i_6 = 1)$ and $((j_2 = 1)$ or $(m_{12} = 0))$ and

$$\left(\exists i = 0, ..., \min\left\{\left\lfloor \frac{m_{13} - 2r}{2} \right\rfloor, m_{23}\right\}\right)\left(\delta\begin{pmatrix} i_1, 1, i_4, i_8, m_{11}, m_{12} + i, m_{13} - 2 \cdot i, m_{14}, \\ m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, r, i \end{pmatrix} = 1\right)$$

Proof: First, suppose that $\gamma\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$. Distinguish two cases:

CASE 1: $i_6 = 0$.

Let $G \in C\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$. Note that

$G \in D\begin{pmatrix} i_1, j_2, i_4, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, 0 \end{pmatrix}$. Therefore, $\delta\begin{pmatrix} i_1, j_2, i_4, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, 0 \end{pmatrix} = 1$.

CASE 2: $i_6 = 1$.

Note that $m_{12} = 0$ or $j_2 = 1$. Also note that in G there are at most $\min\left\{\left\lfloor \frac{m_{13} - 2r}{2} \right\rfloor, m_{23}\right\}$ (2-methyl-propyl)-thorns. Denote their number by i. Let G' be a graph obtained from graph G by replacing each of these thorns by an ethyl thorn. Note that

$G' \in D\begin{pmatrix} i_1, 1, i_4, i_8, m_{11}, m_{12} + i, m_{13} - 2 \cdot i, m_{14}, \\ m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, r, i \end{pmatrix}$, hence 2) holds.

Now, let us prove the opposite implication. Suppose that 1) holds.

Let $G \in D\begin{pmatrix} i_1, j_2, i_4, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, 0 \end{pmatrix}$. Note that $G \in C\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix}$.

Therefore, $\gamma\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$.

Now, suppose that 2) holds. Let $i \in \left\{0, \ldots, \min\left\{\left\lfloor \frac{m_{13} - 2r}{2} \right\rfloor, m_{23}\right\}\right\}$ such that

$\delta\begin{pmatrix} i_1, 1, i_4, i_8, m_{11}, m_{12} + i, m_{13} - 2 \cdot i, m_{14}, \\ m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, r, i \end{pmatrix} = 1$ and let

$G \in D\begin{pmatrix} i_1, 1, i_4, i_8, m_{11}, m_{12} + i, m_{13} - 2 \cdot i, m_{14}, \\ m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, r, i \end{pmatrix}$. Note that all edges of type e_{12} are part of ethyl-thorns in G. Threrefore there are $m_{12} + i$ ethyl-thorns in G. Let G' be a graph obtained from graph G by replacing i of these thorns by (2-methyl-propyl)-thorns. Note that

$C\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix}$, hence $\gamma\begin{pmatrix} j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$.

All the cases are exhausted and the theorem is proved. □

Let C be any cycle and let f be any function $f : V(C) \to \{2,3,4\}$. Denote $\theta(C, f) = (x_{22}, x_{23}, x_{24}, x_{33}, x_{34}, x_{44})$, where

$$x_{ij} = card\{uv \in E(C) : f(u) = i \text{ and } f(v) = j\}, \text{ for each } 2 \le i \le j \le 4.$$

Let us give an auxiliary result.

Lemma 1.5 Let $(x_{22}, x_{23}, x_{24}, x_{33}, x_{34}, x_{44}) \in N_0^6$. There is a cycle C and function f such that $\theta(C, f) = (x_{22}, x_{23}, x_{24}, x_{33}, x_{34}, x_{44})$ if and only if the following holds:

1) $x_{23} + x_{34}, x_{24} + x_{34}$ are even numbers;

2) $x_{22} + x_{23} + x_{24} + x_{33} + x_{34} + x_{44} \ge 3$;

3) $(x_{23} + x_{24} > 0)$ or $(x_{22} = 0)$ or $(x_{23} = x_{24} = x_{33} = x_{34} = x_{44} = 0)$;

4) $(x_{23} = x_{33} = x_{34} = 0)$ or $(x_{24} = x_{34} = x_{44} = 0)$ or $(x_{34} + \min\{x_{23}, x_{24}\} \ge 2)$.

Proof: First, let us prove necessity. Distinguish seven cases:

CASE 1: $x_{23} = x_{24} = x_{33} = x_{34} = x_{44} = 0$.

It is sufficent to take any cycle C with x_{22} vertices and the constant function that assigns number 2 to each vertex.

CASE 2: $x_{22}=x_{23}=x_{33}=x_{34}=0$.

It is sufficent to take cycle C wtih vertices (in order of their apperiance) $v_1,v_2,...,v_{x_{24}+x_{44}}$ and function $f:V(C)\rightarrow\{2,3,4\}$ such that

$$f(v_i)=\begin{cases}2, & i\le x_{24} \text{ and } i \text{ is even}\\ 4, & \text{otherwise}\end{cases}.$$

CASE 3: $x_{23}=x_{33}=x_{34}=0$ and $x_{22}>0$ and $x_{24}+x_{44}>0$.

Note that in graph C (note that here C is not simple if $x_{24}+x_{44}<3$) constructed in the previous case, there is at least one edge incident to vertices w and z such that $f(w)=2$ and $f(z)=4$. Replace this edge by a path $wu_1u_2...u_{x_{22}}z$ and extend the definiton of the function *f* by $f(u_i)=2$ for each $i=1,...,x_{22}$. Graph and function obtained in this way have the required properties.

CASE 4: $x_{22}=x_{24}=x_{34}=x_{44}=0$.

It is sufficent to take cycle C wtih vertices (in order of their apperiance) $v_1,v_2,...,v_{x_{23}+x_{33}}$ and function $f:V(C)\rightarrow\{2,3,4\}$ such that

$$f(v_i)=\begin{cases}2, & i\le x_{23} \text{ and } i \text{ is even}\\ 3, & \text{otherwise}\end{cases}.$$

CASE 5: $x_{24}=x_{34}=x_{44}=0$ and $x_{22}>0$ and $x_{23}+x_{33}>0$.

Note that in graph C defined in the previous case, there is at least one edge incident to vertices w and z such that $f(w)=2$ and $f(z)=3$. Replace this edge by a path $wu_1u_2...u_{x_{22}}z$ and extend the definiton of the function *f* by $f(u_i)=2$ for each $i=1,...,x_{22}$. Graph and function obtained in this way have the required properties.

CASE 6: $x_{34}+\min\{x_{23},x_{24}\}\ge 2$ and $x_{22}=0$.

Distinguish three subcases:

SUBCASE 6.1: $x_{34}\le 2$.

Let cycle C_1 be the cycle wtih vertices (in order of their apperiance) $u_1,u_2,...,u_{(x_{23}-(2-x_{34}))+x_{33}+1},v_1,v_2,...,v_{(x_{24}-(2-x_{34}))+x_{44}+1}$ and define the function $f:V(C)\rightarrow\{2,3,4\}$ by

$$f(v)=\begin{cases}2, & (v=u_i, i \text{ is even and } i\le x_{23}-(2-x_{34})) \text{ or } (v=v_i, i \text{ is even and } i\le x_{24}-(2-x_{34}))\\ 3, & v=u_i \text{ and } (i \text{ is odd or } i>x_{23}-(2-x_{34}))\\ 4, & v=v_i \text{ and } (i \text{ is odd or } i>x_{24}-(2-x_{34}))\end{cases}.$$

Note that cycle C has exactly two edges e_1 and e_2 such that function f ssigns to their endvertices values 3 and 4. Replace exactly $2-x_{34}$ of these edges by a paths of length 2. Denote graph obtained in this way by C and extend the definition of the function *f* by putting $f(v)=2$ for all new vertices. Graph C and function *f* have required properties.

SUBCASE 6.2: $x_{34}>2$ even.

It is sufficent to take cycle C wtih vertices (in order of their apperiance)

$$u_1,u_2,...,u_{x_{23}+x_{33}+1},v_1,v_2,...v_{x_{24}+x_{44}+1},z_1,z_2...,z_{x_{34}-2}$$

and function $f:V(C)\to\{2,3,4\}$ such that

$$f(v)=\begin{cases}2, & (v=u_i, i\le x_{23} \text{ and } i \text{ is even}) \text{ or } (v=v_i, i\le x_{24} \text{ and } i \text{ is even})\\ 3, & (v=u_i \text{ and } (i>x_{23} \text{ or } i \text{ is odd})) \text{ or } (v=z_i, i \text{ is odd})\\ 4, & (v=v_i \text{ and } (i>x_{24} \text{ or } i \text{ is odd})) \text{ or } (v=z_i, i \text{ is even})\end{cases}.$$

SUBCASE 6.3: $x_{34}>2$ odd.

It is sufficent to take cycle C wtih vertices (in order of their apperiance)

$$u_1,u_2,\dots,u_{x_{23}+x_{33}},w,v_1,v_2,\dots v_{x_{24}+x_{44}},z_1,z_2\dots,z_{x_{34}-1}$$

and function $f:V(C)\to\{2,3,4\}$ such that

$$f(v)=\begin{cases}2, & (v=u_i, i\le x_{23}-1 \text{ and } i \text{ is even}) \text{ or } (v=v_i, i\le x_{24}-1 \text{ and } i \text{ is even}) \text{ or } (v=w)\\ 3, & (v=u_i \text{ and } (i>x_{23}-1 \text{ or } i \text{ is odd})) \text{ or } (v=z_i, i \text{ is odd})\\ 4 & (v=v_i \text{ and } (i>x_{24}-1 \text{ or } i \text{ is odd})) \text{ or } (v=z_i, i \text{ is even})\end{cases}.$$

CASE 7: $x_{34}+\min\{x_{23},x_{24}\}\ge 2$ and $x_{22}=0$.

This case can be solved by the complete analogy with cases 3 and 6.

All the cases are exhausted and necessity is proved.

Now, let us prove sufficency. Let the cycle C and function f have the required properties. Denote $N_i=\{v: f(v)=i\}$ for each $i=2,3,4$. Note that $|N_3|=\dfrac{x_{23}+2x_{33}+x_{34}}{2}$, hence indeed $x_{23}+x_{24}$ is an even number. Analogously $|N_4|=\dfrac{x_{24}+x_{34}+2x_{44}}{2}$ implies that $x_{24}+x_{34}$ is an even number. Since C is cycle it must have at least three vertices, so 2) holds. Suppose that 3) doesn't hold. Then N_2 and $N_3\cup N_4$ are nonempty set and $e(N_2,N_3\cup N_4)=\varnothing$, which is a contradiciton, therefore 3) holds. Now, suppose that 4) doesn't hold. Note that N_3 and N_4 are nonempty sets. Distinguish two cases:

CASE 1: $m_{23}+m_{34}<2$.

We have $e(N_3,N_2\cup N_4)<2$, which is a contradiction.

CASE 2: $m_{24}+m_{34}<2$.

We have $e(N_4,N_2\cup N_3)<2$, which is a contradiction.

All the cases are exhausted and the lemma is proved. □

Let us define the function $\phi:\mathbb{Z}^6\to\{0,1\}$ by $\phi(x_{22},x_{23},x_{24},x_{33},x_{34},x_{44})=1$ if and only if $x_{22},x_{23},x_{24},x_{33},x_{34},x_{44}\ge 0$ and they satisfy the conditions 1) – 4) of the last Lemma.

Let us prove:

Lemma1.6 $\delta\begin{pmatrix} j_1,j_2,j_3,j_4,m_{11},m_{12},m_{13},m_{14},\\ m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r,s\end{pmatrix}=1$ if and only if:

1) $m_{11}=0$
2) $n_3,n_4\in N$
3) $n_4=2n_3+n_1$

4) $m_{12} \ge s$
5) There are numbers $m_3, m_4, d_3, d_4, t_3, t_4, q_3, q_4$ such that:
5.1) $d_3 + d_4 = m_{12}$
5.2) $m_3 + 2t_3 + 2t_4 = m_{13}$
5.3) $m_4 + 3q_3 + 3q_4 = m_{14}$
5.4) $m_3 + d_3 + 2t_3 + t_4 + q_3 = n_3$
5.5) $t_3 + t_4 \ge r$
5.6) $t_3 + t_4 = r$ or $j_3 < 2$
5.7) $m_3 = m_4 = 0$ or $j_1 > 0$
5.8) $d_3 = d_4 = 0$ or $j_2 > 0$
5.9) $t_3 = t_4 = 0$ or $j_3 > 0$
5.10) $q_3 = q_4 = 0$ or $j_4 > 0$
5.11) $x_{22}, x_{23}, x_{24}, x_{33}, x_{34}, x_{44} \ge 0$
5.12) $x_{23} + x_{34}, x_{24} + x_{34}$ are even numbers;
5.13) $x_{22} + x_{23} + x_{24} + x_{33} + x_{34} + x_{44} \ge 3$;
5.14) $(x_{23} + x_{24} > 0)$ or $(x_{22} = 0)$ or $(x_{23} = x_{24} = x_{33} = x_{34} = x_{44} = 0)$;
5.15) $(x_{23} = x_{33} = x_{34} = 0)$ or $(x_{24} = x_{34} = x_{44} = 0)$ or $(x_{34} + \min\{x_{23}, x_{24}\} \ge 2)$,
where
$x_{22} = m_{22}$; $x_{23} = m_{23} - d_3$; $x_{24} = m_{24} - d_4$; $x_{33} = m_{33} - t_3$; $x_{34} = m_{34} - q_3 - t_4$; $x_{44} = m_{44} - q_4$;
$n_1 = m_{12} + m_{13} + m_{14}$; $n_3 = \dfrac{m_{13} + m_{23} + 2m_{33} + m_{34}}{3}$ and $n_4 = \dfrac{m_{14} + m_{24} + m_{34} + 2m_{44}}{4}$.

Proof: First, suppose that $\delta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$. Let $G \in D\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix}$. Obviously, 1), 2) and 4) hold and 3) holds because G is monocyclical graph . Denote by $m_j, j = 3,4$, the number of methyl-thorns adjacent to vertex of degree j; by $d_j, j = 3,4,$ the number of ethyl-thorns adjacent to vertex of degree j; by $t_j, j = 3,4,$ the number of isopropyl-thorns adjacent to vertex of degree j; and by $q_j, j = 3,4,$ the number of (1,1-dimethyl-ethyl)-thorns adjacent to vertex of degree j. It can be easily checked that relations 5.1) – 5.10) are satisfied. Note that x_{ij} is in fact number of edges in $C(G)$ that connect vertices of degrees (in G) i and j. Therefore, 5.11) holds and relations 5.12) – 5.15) are implied by the last Lemma.
Now, let us prove the opposite inequality. Suppose that relations 1)-4) hold. From the relations 5.12) – 5.15), it follows that there is a cycle C and function f such that $\theta(C, f) = (x_{22}, x_{23}, x_{24}, x_{33}, x_{34}, x_{44})$. Denote, as in the previos lemma, $N_i = \{v : f(v) = i\}$ for each $i = 2,3,4$. Note that

$$\left|N_3\right| = \frac{x_{23} + 2x_{33} + x_{34}}{2} = \frac{m_{23} - d_3 + 2m_{33} - 2t_3 + m_{34} - q_3 - t_4}{2} =$$

$$= \frac{3n_3 - m_{13} - d_3 - q_3 - 2t_3 - t_4}{2} =$$

$$= \frac{3\left(m_3 + d_3 + 2t_3 + t_4 + q_3\right) - \left(m_3 + 2t_3 + 2t_4\right) - d_3 - q_3 - 2t_3 - t_4}{2}$$

$$= m_3 + d_3 + q_3 + t_3$$

and that

$$\left|N_4\right| = \frac{m_{24} - d_4 + m_{34} - t_4 - q_3 + 2m_{44} - 2q_4}{2}$$

$$= \frac{4n_4 - m_{14} - d_4 - t_4 - q_3 - 2q_4}{2}$$

$$= \frac{4 \cdot \frac{n_1 - n_3}{2} - m_{14} - d_4 - t_4 - q_3 - 2q_4}{2}$$

$$= \frac{2\left(m_{12} + m_{13} + m_{14}\right) - 2\left(m_3 + d_3 + 2t_3 + t_4 + q_3\right) - m_{14} - d_4 - t_4 - q_3 - 2q_4}{2} =$$

$$= \frac{2\left(d_3 + d_4\right) + 2\left(m_3 + 2t_3 + 2t_4\right) + \left(m_4 + 3q_3 + 3q_4\right) - 2m_3 - 2d_3 - d_4 - 4t_3 - 3t_4 - 3q_3 - 2q_4}{2}$$

$$= \frac{m_4 + d_4 + q_4 + t_4}{2}$$

Note that $x_{24} + x_{34}$ is nonnegative even number, hence $\left|N_4\right|$ is non-negative integer. Add to each vertex in N_3 exactly one thorn in such way that there among added thorns there are m_3 methyl-thorns, d_3 ethyl-thorns, t_3 isopropyl-thorns and q_3 (1,1-dimethyl-ethyl)-thorns and add to each vertex in N_4 exactly two thorns in such way that there among added thorns there are m_4 methyl-thorns, d_4 ethyl-thorns, t_4 isopropyl-thorns and q_4 (1,1-dimethyl-ethyl)-thorns. Denote graph obtained in this way by G. Simple calculation shows that $G \in D\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$. Therefore, $\delta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$, Which proves the claim. □

Let us observe the relations 5.1) - 5.4). They can be observed as the system of four equations with unknowns $m_3, m_4, d_3, d_4, t_3, t_4, q_3$ and q_4. Solving this system, we get a four-parameter solution

$$
\begin{aligned}
m_3 &= m_{13} - 2x - 2y \\
m_4 &= m_{14} - 3n_3 + 3m_{13} - 3y + 3u - 3z \\
d_3 &= u \\
d_4 &= m_{12} - u \\
t_3 &= x \\
t_4 &= y \\
q_3 &= n_3 - m_{13} + y - u \\
q_4 &= z
\end{aligned}
\qquad (\#)
$$

Now, we can reformulate (by a straight forward calculation) the last theorem:

Lemma 1.7 $\delta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if:

1) $m_{11} = 0$
2) $n_3, n_4 \in N$
3) $n_4 = 2n_3 + n_1$
4) $m_{12} \geq s$
5) $m_{24} - m_{12} + m_{34} - n_3 + m_{13}$ is an even number
5) There are integers x, y, z, u such that:
5.1) $x + y \geq r$
5.2) $x \geq 0$
5.3) $y \geq 0$
5.4) $z \geq 0$
5.5) $u \geq 0$
5.6) $m_{13} - 2x - 2y \geq 0$
5.7) $n_3 - m_{13} + y - u \geq 0$
5.8) $m_{14} - 3n_3 + 3m_{13} - 3y + 3u - 3z \geq 0$
5.9) $m_{12} - u \geq 0$
5.10) $x + y = r$ or $j_3 < 2$
5.11) $m_{13} - 2x - 2y = 0$ or $j_1 > 0$
5.12) $m_{14} - 3n_3 + 3m_{13} - 3y + 3u - 3z = 0$ or $j_1 > 0$
5.13) $u = 0$ or $j_2 > 0$
5.14) $m_{12} - u = 0$ or $j_2 > 0$
5.15) $x = 0$ or $j_3 > 0$
5.16) $y = 0$ or $j_3 > 0$
5.17) $n_3 - m_{13} + y - u = 0$ or $j_4 > 0$
5.18) $z = 0$ or $j_4 > 0$
5.19) $m_{22} + m_{23} + m_{24} + m_{33} + m_{34} - m_{12} - n_3 + m_{13} - 3 + u - x - 2y - z \geq 0$

5.20) $(m_{22}=0)$ or $(m_{23}+m_{24}-m_{12}=0)$ or

$\left(x=m_{33};\ y=\frac{1}{2}(m_{34}-n_3+m_{13}+u);z=m_{44};u=m_{23};m_{24}-m_{12}=m_{23}\right)$

5.21) $\left(u=m_{23};\ x=m_{33};\ y=\frac{1}{2}(m_{34}-n_3+m_{13}+u)\right)$ or

$\left(y=\frac{1}{2}(m_{34}-n_3+m_{13}+u);\ u=m_{12}-m_{24};\ z=m_{44}\right)$ or

$\left((m_{23}+m_{34}-n_3+m_{13}-2-2y\geq 0)\right.$ and $\left.(\mathrm{m}_{24}-m_{12}+m_{34}-n_3+m_{13}-2+2u-2y\geq 0)\right)$

5.22) $m_{23}-u\geq 0$

5.23) $m_{24}-m_{12}+u\geq 0$

5.24) $m_{33}-x\geq 0$

5.25) $m_{34}-n_3+m_{13}-2y+u\geq 0$

5.26) $m_{44}-z\geq 0$

where

$n_1=m_{12}+m_{13}+m_{14}$; $n_3=\frac{m_{13}+m_{23}+2m_{33}+m_{34}}{3}$ and $n_4=\frac{m_{14}+m_{24}+m_{34}+2m_{44}}{4}$.□

It can be easily checked that

Lemma 1.8. Let $x,y,z,u\in Z$ be the numbers that satisfy relation 5.1) – 5.26) and let $\max\{0,r-y\}\leq x'\leq x$. Then numbers x',y,z,u also satisfy relations 5.1) – 5.26) except possibly relations 5.10), 5.11) and 5.15). □

Lemma 1.9. Let $x,y,z,u\in Z$ be the numbers that satisfy relation 5.1) – 5.26) and let Then numbers $x,y,0,u$ also satisfy relations 5.1) – 5.26) except possibly relations 5.10) and 5.15). □

Using Lemmas 1.8 and 1.9 the relation 5.21 from the Lemma 1.7 can be reformulated as

5.21') $\left(u=m_{23};\ x=m_{33};\ y=\frac{1}{2}(m_{34}-n_3+m_{13}+u);\ z=0\right)$ or

$\left(x=r-\frac{1}{2}(m_{34}-n_3+m_{13}+u);\ y=\frac{1}{2}(m_{34}-n_3+m_{13}+u);\ u=m_{12}-m_{24};\ z=m_{44}\right)$ or

$\left(x=0;\ y=\frac{1}{2}(m_{34}-n_3+m_{13}+u);\ u=m_{12}-m_{24};\ z=m_{44}\right)$ or

$\left(x=\frac{m_{13}}{2}-y;\ y=\frac{1}{2}(m_{34}-n_3+m_{13}+u);\ u=m_{12}-m_{24};\ z=m_{44}\right)$

$\left((m_{23}+m_{34}-n_3+m_{13}-2-2y\geq 0)\right.$ and $\left.(\mathrm{m}_{24}-m_{12}+m_{34}-n_3+m_{13}-2+2u-2y\geq 0)\right)$

Let function $\varepsilon:\{0,1\}^2\times\{0,1,2\}\times\{0,1\}\times N_0^{16}\to\{0,1\}$ be given by $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, x, y, z, u \end{pmatrix}=1$ if and only if "There is a thorny cycle G such that $\mu(G)=\begin{pmatrix} m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44} \end{pmatrix}$ with the following thorn allowed: methyl-thorn (if $j_1=1$); ethyl-thorn (if $j_2=1$); isopropyl-thorn (if $j_3>1$); and (1,1-dimethyl-ethyl)-thorn (if $j_4=1$) there are at least r isopropyl-thorns (if $i_4\neq 2$) or there are exactly r isopropyl-thorns (if $i_4=2$); there are at least s ethyl-thorns; there are u ethyl-thorns adjacent to vertices of degree 3; x isopropyl-thorns adjacent with vertices of degree 3; y isopropyl-thorns adjacent to vertices of degree 4; z (1,1-dimethyl-ethyl)-thorns adjacent with vertices of degree 4; and there are n_3 vertices of degree 3. Denote the set of all graphs by with these required properties by $E\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, n_3, r, s, x, y, u, v \end{pmatrix}$.

Now, we can rewrite Lemma 1.7 as

Lemma 1.10 $\delta\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix}=1$ if and only if $m_{12}\geq s$; $m_{11}=0$; $n_3, n_4\in N$; $n_1=2n_4+n_3$ and one of the following claims hold:

1) $m_{34}-n_3+m_{13}+m_{23}=0(\bmod 2)$ and $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ m_{33}, (m_{34}-n_3+m_{13}+m_{23})/2, m_{44}, m_{23} \end{pmatrix}=1$

2) $j_1=1$ and one of the following claims hold

2.1) $m_{34}-n_3+m_{13}+m_{23}=0(\bmod 2)$ and $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ m_{33}, (m_{34}-n_3+m_{13}+m_{23})/2, 0, m_{23} \end{pmatrix}=1$

2.2) $m_{34}-n_3+m_{13}+m_{12}-m_{24}=0(\bmod 2)$ and one of the following claims hold

2.2.1) $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ 0, (m_{34}-n_3+m_{13}+m_{12}-m_{24})/2, m_{44}, m_{12}-m_{24} \end{pmatrix}=1$

2.2.2) $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, r-(m_{34}-n_3+m_{13}+m_{12}-m_{24})/2, \\ (m_{34}-n_3+m_{13}+m_{12}-m_{24})/2, m_{44}, m_{12}-m_{24} \end{pmatrix}=1$

3) $j_1=0$ and one of the following claims hold

3.1) $m_{34} - n_3 + m_{13} + m_{23} = 0 \pmod 2$ and $m_{14} = 0 \pmod 3$ and

$$\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, m_{33}, \left(m_{34} - n_3 + m_{13} + m_{23}\right)/2, \\ m_{14}/3 - n_3 + m_{13} - \left(m_{34} - n_3 + m_{13} + m_{23}\right)/2 + m_{23}, m_{23} \end{pmatrix} = 1$$

3.2) $m_{34} - n_3 + m_{13} + m_{12} - m_{24} + m_{23} = 0 \pmod 2$ and $m_{13} = 0 \pmod 2$ and

$$\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, m_{13}/2 - \left(m_{34} - n_3 + m_{13} + m_{12} - m_{24} + m_{23}\right)/2, \\ \left(m_{34} - n_3 + m_{13} + m_{12} - m_{24} + m_{23}\right)/2, m_{44}, m_{12} - m_{24} \end{pmatrix} = 1$$

4) The following holds:

4.1) $m_{24} - m_{12} + m_{34} - n_3 + m_{13}$ is an even number

4.2) $(m_{22} = 0)$ or $(m_{23} + m_{24} - m_{12} = 0)$

4.3) There are integers x, y, z, u such that:

4.3.1) $x + y \geq r$

4.3.2) $x \geq 0$

4.3.3) $y \geq 0$

4.3.4) $z \geq 0$

4.3.5) $u \geq 0$

4.3.6) $m_{13} - 2x - 2y \geq 0$

4.3.7) $n_3 - m_{13} + y - u \geq 0$

4.3.8) $m_{14} - 3n_3 + 3m_{13} - 3y + 3u - 3z \geq 0$

4.3.9) $m_{12} - u \geq 0$

4.3.10) $x + y = r$ or $j_3 < 2$

4.3.11) $m_{13} - 2x - 2y = 0$ or $j_1 > 0$

4.3.12) $m_{14} - 3n_3 + 3m_{13} - 3y + 3u - 3z = 0$ or $j_1 > 0$

4.3.13) $u = 0$ or $j_2 > 0$

4.3.14) $m_{12} - u = 0$ or $j_2 > 0$

4.3.15) $x = 0$ or $j_3 > 0$

4.3.16) $y = 0$ or $j_3 > 0$

4.3.17) $n_3 - m_{13} + y - u = 0$ or $j_4 > 0$

4.3.18) $z = 0$ or $j_4 > 0$

4.3.19) $m_{22} + m_{23} + m_{24} + m_{33} + m_{34} - m_{12} - n_3 + m_{13} - 3 + u - x - 2y - z \geq 0$

4.3.20) $m_{23} + m_{34} - n_3 + m_{13} - 2 - 2y \geq 0$

4.3.21) $m_{24} - m_{12} + m_{34} - n_3 + m_{13} - 2 + 2u - 2y \geq 0$

4.3.22) $m_{23} - u \geq 0$

4.3.23) $m_{24} - m_{12} + u \geq 0$

4.3.24) $m_{33} - x \geq 0$

4.3.25) $m_{34} - n_3 + m_{13} - 2y + u \geq 0$

4.3.26) $m_{44} - z \geq 0$

where

$n_1 = m_{12} + m_{13} + m_{14}$; $n_3 = \dfrac{m_{13} + m_{23} + 2m_{33} + m_{34}}{3}$ and $n_4 = \dfrac{m_{14} + m_{24} + m_{34} + 2m_{44}}{4}$.□

Let function $\zeta : \{0,1\}^2 \times \{0,1,2\} \times \{0,1\} \times N_0^{12} \to \{0,1\}$ be given by $\zeta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if they satisfy the condition 4) of the last Lemma. Now, we can reformulate the last Lemma as:

Theorem 1.11 $\delta\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if $m_{12} \geq s$; $m_{11} = 0$; $n_3, n_4 \in N$ and one of the following claims hold:

1) $m_{34} - n_3 + m_{13} + m_{23} = 0 \pmod 2$ and $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ m_{33}, (m_{34} - n_3 + m_{13} + m_{23})/2, m_{44}, m_{23} \end{pmatrix} = 1$

2) $j_1 = 1$ and one of the following claims hold

2.1) $m_{34} - n_3 + m_{13} + m_{23} = 0 \pmod 2$ and $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ m_{33}, (m_{34} - n_3 + m_{13} + m_{23})/2, 0, m_{23} \end{pmatrix} = 1$

2.2) $m_{34} - n_3 + m_{13} + m_{12} - m_{24} = 0 \pmod 2$ and one of the following claims hold

2.2.1) $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ 0, (m_{34} - n_3 + m_{13} + m_{12} - m_{24})/2, m_{44}, m_{12} - m_{24} \end{pmatrix} = 1$

2.2.2) $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, r - (m_{34} - n_3 + m_{13} + m_{12} - m_{24})/2, \\ (m_{34} - n_3 + m_{13} + m_{12} - m_{24})/2, m_{44}, m_{12} - m_{24} \end{pmatrix} = 1$

3) $j_1 = 0$ and one of the following claims hold

3.1) $m_{34} - n_3 + m_{13} + m_{23} = 0 \pmod 2$ and $m_{14} = 0 \pmod 3$ and

$$\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, m_{33}, (m_{34} - n_3 + m_{13} + m_{23})/2, \\ m_{14}/3 - n_3 + m_{13} - (m_{34} - n_3 + m_{13} + m_{23})/2 + m_{23}, m_{23} \end{pmatrix} = 1$$

3.2) $m_{34} - n_3 + m_{13} + m_{12} - m_{24} + m_{23} = 0 \pmod 2$ and $m_{13} = 0 \pmod 2$ and

$$\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, m_{13}/2 - (m_{34} - n_3 + m_{13} + m_{12} - m_{24} + m_{23})/2, \\ (m_{34} - n_3 + m_{13} + m_{12} - m_{24} + m_{23})/2, m_{44}, m_{12} - m_{24} \end{pmatrix} = 1$$

4) $\zeta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$□

Straight forward analysis shows that

Theorem 1.12 $\varepsilon\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, x, y, z, u \end{pmatrix} = 1$ if and only if the following holds:

1) $m_{12} \geq s$

2) $m_{11} = 0$

3) $n_3, n_4 \in N$

4) $n_1 = 2n_4 + n_3$

5) $m_3, m_4, d_3, d_4, t_3, t_4, q_3, q_4 \geq 0$

6) $t_3 + t_4 = r$ or $j_3 < 2$

7) $m_3 = m_4 = 0$ or $j_1 > 0$

8) $d_3 = d_4 = 0$ or $j_2 > 0$

9) $t_3 = t_4 = 0$ or $j_3 > 0$

10) $q_3 = q_4 = 0$ or $j_4 > 0$

11) $d_3 + d_4 \geq s$

12) $\phi\left(m_{22}, m_{23} - d_3, m_{24} - d_4, m_{33} - t_3, m_{34} - t_4 - q_3, m_{44} - q_4\right) = 1$

where n_1, n_3 and n_4 are defined as in the last theorem; and $m_3, m_4, d_3, d_4, t_3, t_4, q_3$ and q_4 are defined as in (#).□

We have

Theorem 1.13 $\zeta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if

1) $m_{24} - m_{12} + m_{34} - n_3 + m_{13}$ is an even number

2) $(m_{22} = 0)$ or $(m_{23} + m_{24} - m_{12} > 0)$

3) There are integers x, y, u such that:

3.1) $x + y \geq r$

3.2) $x \geq 0$

3.3) $y \geq 0$

3.4) $u \geq 0$

3.5) $m_{13} - 2x - 2y \geq 0$

3.6) $a + y - u \geq 0$

3.7) $m_{12} - u \geq 0$

3.8) $m_{23} - u \geq 0$

3.9) $f + u \geq 0$

3.10) $m_{33} - x \geq 0$

3.11) $g - 2y + u \geq 0$

3.12) $d - 2y \geq 0$

3.13) $e + 2u - 2y \geq 0$
3.14) $x + y = r$ or $j_3 < 2$
3.15) $m_{13} - 2x - 2y = 0$ or $j_1 > 0$
3.16) $u = 0$ or $j_2 > 0$
3.17) $m_{12} - u = 0$ or $j_2 > 0$
3.18) $x = 0$ or $j_3 > 0$
3.19) $y = 0$ or $j_3 > 0$
3.20) $a + y - u = 0$ or $j_4 > 0$
3.21) There is an integer z such that
3.21.1) $z \geq 0$
3.21.2) $b - 3y + 3u - 3z \geq 0$
3.21.3) $m_{44} - z \geq 0$
3.21.4) $c + u - x - 2y - z \geq 0$
3.21.5) $b - 3y + 3u - 3z = 0$ or $j_1 > 0$
3.21.6) $z = 0$ or $j_4 > 0$
where

$$a = n_3 - m_{13}$$
$$b = m_{14} - 3n_3 + m_{13}$$
$$c = m_{22} + m_{23} + m_{24} + m_{33} + m_{34} + m_{44} - m_{12} - n_3 + m_{13} - 3$$
$$d = m_{23} + m_{34} - n_3 + m_{13} - 2$$
$$e = m_{24} - m_{12} + m_{34} - n_3 + m_{13} - 2$$
$$f = m_{24} - m_{12}$$
$$g = m_{34} - n_3 + m_{13}$$

and n_1, n_3 and n_4 are defined as in the last theorem. □

Note that condition 3.21) can be rewritten as
3.21') There is an integer z such that
3.21'.1) $z \geq 0$
3.21'.2) $z \leq \left\lfloor \frac{b}{3} - y + u \right\rfloor$
3.21'.3) $z \leq m_{44}$
3.21'.4) $z \leq c + u - x - 2y$
3.21'.5) $z = \frac{b}{3} - y + u$ or $j_1 > 0$
3.21'.6) $z = 0$ or $j_4 > 0$

Another reformulation of this conditions is the following:

1) $0 \leq \frac{b}{3} - y + u$
2) $0 \leq c + u - x - 2y$

3) $\frac{b}{3} - y + u \le m_{44}$ or $j_1 > 0$

4) $\frac{b}{3} - y + u \le c + u - x - 2y$ or $j_1 > 0$

5) $(b = 0 (\bmod 3))$ or $j_1 > 0$

6) $\frac{b}{3} - y + u = 0$ or $j_1 > 0$ or $j_4 > 0$

This allows us to rewrite Theorem 1.13 as

Theorem 1.14 $\zeta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if

1) $m_{24} - m_{12} + m_{34} - n_3 + m_{13}$ is an even number

2) $(m_{22} = 0)$ or $(m_{23} + m_{24} - m_{12} > 0)$

3) $(b = 0 (\bmod 3))$ or $j_1 > 0$

4) There are integers y, u such that:

4.1) $y \ge 0$

4.2) $u \ge 0$

4.3) $a + y - u \ge 0$

4.4) $m_{12} - u \ge 0$

4.5) $m_{23} - u \ge 0$

4.6) $f + u \ge 0$

4.7) $g - 2y + u \ge 0$

4.8) $d - 2y \ge 0$

4.9) $e + 2u - 2y \ge 0$

4.10) $u = 0$ or $j_2 > 0$

4.11) $m_{12} - u = 0$ or $j_2 > 0$

4.12) $y = 0$ or $j_3 > 0$

4.13) $a + y - u = 0$ or $j_4 > 0$

4.14) $u - y \ge -h$

4.15) $u - y \le i$ or $j_1 > 0$

4.16) $h - y + u = 0$ or $j_1 > 0$ or $j_4 > 0$

4.17) There is an integer x such that

4.17.1) $x + y \ge r$

4.17.2) $x \ge 0$

4.17.3) $m_{13} - 2x - 2y \ge 0$

4.17.4) $m_{33} - x \ge 0$

4.17.5) $u - x - 2y \ge -c$

4.17.6) $x + y = r$ or $j_3 < 2$

4.17.7) $m_{13} - 2x - 2y = 0$ or $j_1 > 0$

4.17.8) $x = 0$ or $j_3 > 0$

4.17.9) $x + y \le j$ or $j_1 > 0$
where

$$h = \lfloor b/3 \rfloor$$
$$i = m_{44} - h$$
$$j = c - h$$

and $n_1, n_3, n_4, a, b, c, d, e, f$ and g are defined as before.□

Note that condition 4.17) of the last theorem can be reformulated as
4.17') There is an integer x such that
4.17'.1) $x \ge r - y$
4.17'.2) $x \ge 0$
4.17'.3) $x \le k - y$
4.17'.4) $x \le m_{33}$
4.17'.5) $x \le u - 2y + c$
4.17'.6) $x = r - y$ or $j_3 < 2$
4.17'.7) $x = k - y$ or $j_1 > 0$
4.17'.8) $x = 0$ or $j_3 > 0$
4.17'.9) $x \le j - y$ or $j_1 > 0$
4.17'.10) $m_{13} = 0 (\mathrm{mod}\, 2)$ or $j_1 > 0$
where $k = \lfloor m_{13}/2 \rfloor$
This can be further reformulated as
1) $r \le k$
2) $y \ge l$
3) $y - u \le m$
4) $y \le k$
5) $u - 2y \ge -c$
6) $y \le r$ or $j_3 \ne 2$
7) $y \ge o$ or $j_1 > 0$
8) $y - u \le p$ or $j_1 > 0$
9) $y \ge r$ or $j_3 > 0$
10) $k \le j$ or $j_1 > 0$
11) $y = k$ or $j_1 > 0$ or $j_3 > 0$
12) $k = r$ or $j_1 > 0$ or $j_3 \ne 2$
13) $m_{13} = 0 (\mathrm{mod}\, 2)$ or $j_1 > 0$

where $l = r - m_{33}$; $m = c - r$; $o = k - m_{33}$ and $p = c - k$.

Now, theorem 1.14 can be reformulated as:
Theorem 1.15 $\zeta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if
1) $m_{24} - m_{12} + m_{34} - n_3 + m_{13}$ is an even number

2) $\left(m_{22}=0\right)$ or $\left(m_{23}+m_{24}-m_{12}>0\right)$

3) $\left(b=0\left(\bmod 3\right)\right)$ or $j_1>0$

4) $m_{13}=0\left(\bmod 2\right)$ or $j_1>0$

5) $k\le j$ or $j_1>0$

6) $r\le k$

7) $k=r$ or $j_1>0$ or $j_3\ne 2$

8) There is an integer y such that:

8.1) $y\ge 0$

8.2) $d-2y\ge 0$

8.3) $y=0$ or $j_3>0$

8.4) $y\ge l$

8.5) $y\le k$

8.6) $y\le r$ or $j_3\ne 2$

8.7) $y\ge o$ or $j_1>0$

8.8) $y\ge r$ or $j_3>0$

8.9) $y=k$ or $j_1>0$ or $j_3>0$

8.10) There is an integer u such that

8.10.1) $u\ge 0$

8.10.2) $a+y-u\ge 0$

8.10.3) $m_{12}-u\ge 0$

8.10.4) $m_{23}-u\ge 0$

8.10.5) $f+u\ge 0$

8.10.6) $g-2y+u\ge 0$

8.10.7) $e+2u-2y\ge 0$

8.10.8) $u=0$ or $j_2>0$

8.10.9) $m_{12}-u=0$ or $j_2>0$

8.10.11) $a+y-u=0$ or $j_4>0$

8.10.12) $u-y\ge -h$

8.10.13) $u-y\le i$ or $j_1>0$

8.10.14) $h-y+u=0$ or $j_1>0$ or $j_4>0$

8.10.15) $y-u\le m$

8.10.16) $u-2y\ge -c$

8.10.17) $y-u\le p$ or $j_1>0$

where $n_1,n_3,n_4,a,b,c,d,e,f,g,h,i,j,k,l,m,o$ and p are defined as before.□

Further, this can be reformulated as

Theorem 1.16 $\zeta\begin{pmatrix} j_1,j_2,j_3,j_4,m_{11},m_{12},m_{13},m_{14}, \\ m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r,s \end{pmatrix}=1$ if and only if

1) $m_{24}-m_{12}+m_{34}-n_3+m_{13}$ is an even number

2) $(m_{22} = 0)$ or $(m_{23} + m_{24} - m_{12} > 0)$
3) $(b = 0(\bmod 3))$ or $j_1 > 0$
4) $m_{13} = 0(\bmod 2)$ or $j_1 > 0$
5) $k \le j$ or $j_1 > 0$
6) $r \le k$
7) $m_{12} = 0$ or $j_2 > 0$
8) $a = -h$ or $j_1 > 0$ or $j_4 > 0$
9) $0 \ge r$ or $j_3 > 0$
10 $k = 0$ or $j_1 > 0$ or $j_3 > 0$
11) $k = r$ or $j_1 > 0$ or $j_3 \ne 2$
12) There is an integer y such that:
12.1) $y \ge fd$
12.2) $y \le fc$
12.3) $y \ge o$ or $j_1 > 0$
12.4) $y = 0$ or $j_3 > 0$
12.5) $y \le r$ or $j_3 \ne 2$
12.6) There is an integer u such that
12.6.1) $u \ge fa$
12.6.2) $u \le a + y$
12.6.3) $u \le fee$
12.6.4) $u \ge fb + 2y$
12.6.5) $u \ge ff + y$
12.6.6) $u \le i + y$ or $j_1 > 0$
12.6.7) $u \ge -p + y$ or $j_1 > 0$
12.6.8) $u = 0$ or $j_2 > 0$
12.6.9) $u = a + y$ or $j_4 > 0$
where

$$fa = \max\{-f, 0\}$$
$$fb = \max\{-c, -g\}$$
$$fc = \min\{\lfloor d/2 \rfloor, k\}$$
$$fd = \max\{0, l\}$$
$$fee = \min\{m_{12}, m_{23}\}$$
$$ff = \max\{\lceil -e/2 \rceil, -h, -m\}$$

and where $n_1, n_3, n_4, a, b, c, d, e, f, g, h, i, j, k, l, m, o$ and p are defined as before.□

The condition 12.6) of the last theorem can be reformulated as:
1) $fa \le fe$
2) $y \ge fa - a$
3) $y \le fe - ff$

4) $ff \leq a$

5) $y \leq \left\lfloor \frac{fe - fb}{2} \right\rfloor$

6) $y \leq a - fb$

7) $y \geq fg$ or $j_1 > 0$

8) $ff \leq i$ or $j_1 > 0$

9) $y \leq fh$ or $j_1 > 0$

10) $0 \leq fe$ or $j_2 > 0$

11) $y \geq -a$ or $j_2 > 0$

12) $0 \geq fa$ or $j_2 > 0$

13) $y \leq fi$ or $j_2 > 0$

14) $y \leq fj$ or $j_4 > 0$

15) $a \leq i$ or $j_1 > 0$ or $j_4 > 0$

16) $y \geq -i$ or $j_1 > 0$ or $j_2 > 0$

17) $y = -a$ or $j_2 > 0$ or $j_4 > 0$

where

$$fg = fa - i$$
$$fh = i - fb$$
$$fi = \min\left\{-ff, \lfloor -fb/2 \rfloor\right\}$$
$$fj = fe - a$$

This gives us another reformulation of the last theorem:

Theorem 1.16 $\zeta\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix} = 1$ if and only if

1) $m_{24} - m_{12} + m_{34} - n_3 + m_{13}$ is an even number

2) $(m_{22} = 0)$ or $(m_{23} + m_{24} - m_{12} > 0)$

3) $(b = 0 \pmod 3)$ or $j_1 > 0$

4) $m_{13} = 0 \pmod 2$ or $j_1 > 0$

5) $k \leq j$ or $j_1 > 0$

6) $r \leq k$

7) $m_{12} = 0$ or $j_2 > 0$

8) $a = -h$ or $j_1 > 0$ or $j_4 > 0$

9) $0 \geq r$ or $j_3 > 0$

10 $k = 0$ or $j_1 > 0$ or $j_3 > 0$

11) $k = r$ or $j_1 > 0$ or $j_3 \neq 2$

12) $fa \leq fee$

13) $ff \leq a$

14) $ff \leq i$ or $j_1 > 0$

15) $0 \leq fee$ or $j_2 > 0$

16) $0 \geq fa$ or $j_2 > 0$
17) $a \leq i$ or $j_1 > 0$ or $j_4 > 0$
18) There is an integer y such that
18.1) $y \geq fk$
18.2) $y \leq fl$
18.3) $y = 0$ or $j_3 > 0$
18.4) $y \leq r$ or $j_3 \neq 2$
18.5) $y \leq fh$ or $j_1 > 0$
18.6) $y \geq fg$ or $j_1 > 0$
18.7) $y \geq o$ or $j_1 > 0$
18.8) $y \geq -a$ or $j_2 > 0$
18.9) $y \leq fi$ or $j_2 > 0$
18.10) $y \leq fj$ or $j_4 > 0$
18.11) $y \geq -i$ or $j_1 > 0$
18.12) $y = -a$ or $j_2 > 0$ or $j_4 > 0$

Using these theorems we can make an efficient algorithm that solves our problem. An algorithm consists of 7 functions: TestA, TestB, TestC, TestD, TestE, TestF and TestG. First the function TestA is invoked and than it invokes TestB and so on. At the end (in the time that is proportional or less then the product of $\min\{m_{12}, m_{13}, m_{23}\} \cdot \{m_{13}/2, m_{23}\}$) we get a required solution. Here is the pseudocode of our algorithm:

TestG (Input data: $j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, n_3, r, s$)
1) If $m_{24} - m_{12} + m_{34} - n_3 + m_{13} = 0 \pmod 2$ then return 0
2) If $m_{22} > 0$ and $m_{23} + m_{24} - m_{12} \leq 0$ then return 0
3) Let $a, b, c, d, e, f, g, h, i, j, k, l, m, o, p, fa, fb, fc, fd, fee, ff, fg, fh, fi, fj, fk$ and fl be defined as before
4) If at least one of the following conditions in not satisfied return 0
4.1) $m_{24} - m_{12} + m_{34} - n_3 + m_{13}$ is an even number
4.2) $(m_{22} = 0)$ or $(m_{23} + m_{24} - m_{12} > 0)$
4.3) $(b = 0 \pmod 3)$ or $j_1 > 0$
4.4) $m_{13} = 0 \pmod 2$ or $j_1 > 0$
4.5) $k \leq j$ or $j_1 > 0$
4.6) $r \leq k$
4.7) $m_{12} = 0$ or $j_2 > 0$
4.8) $a = -h$ or $j_1 > 0$ or $j_4 > 0$
4.9) $0 \geq r$ or $j_3 > 0$
4.10 $k = 0$ or $j_1 > 0$ or $j_3 > 0$
4.11) $k = r$ or $j_1 > 0$ or $j_3 \neq 2$
4.12) $fa \leq fee$
4.13) $ff \leq a$

4.14) $ff \le i$ or $j_1 > 0$
4.15) $0 \le fee$ or $j_2 > 0$
4.16) $0 \ge fa$ or $j_2 > 0$
4.17) $a \le i$ or $j_1 > 0$ or $j_4 > 0$
5) $dm = fk$
6) If $j_3 = 0$ then $dm = \max\{dm, 0\}$
7) If $j_1 = 0$ then $dm = \max\{dm, fg, o, -i\}$
8) If $j_2 = 0$ then $dm = \max\{dm, -a\}$
9) $gm = fl$
10) If $j_3 = 0$ then $gm = \min\{gm, 0\}$
11) If $j_3 = 2$ then $gm = \min\{gm, r\}$
12) If $j_2 = 0$ then $gm = \min\{gm, fi\}$
13) If $j_4 = 0$ then $gm = \min\{gm, fj\}$
14) If $(j_2 = 0 \text{ and } j_4 = 0)$ then $gm = \min\{gm, -a\}$
15) If $j_1 = 0$ then $gm = \min\{gm, fh\}$
16) If $dm \le gm$ then return 1
17) return 0

TestF (Input data: $x_{22}, x_{23}, x_{24}, x_{33}, x_{34}, x_{44}$)
1) If $x_{22}, x_{23}, x_{24}, x_{33}, x_{34}, x_{44} \ge 0$ and $x_{23} + x_{34} = 0 \pmod 2$ and $x_{24} + x_{34} \ge 0 \pmod 2$ and $x_{22} + x_{23} + x_{24} + x_{33} + x_{34} + x_{44} \ge 3$ and $((x_{23} + x_{24} > 0)$ or $(x_{22} = 0)$ or $(x_{22} = x_{23} = x_{24} = x_{33} = x_{34} = x_{44} = 0))$ and $((x_{23} = x_{33} = x_{34} = 0)$ or $(x_{24} = x_{34} = x_{44} = 0)$ or $(x_{34} + \min\{x_{23}, x_{24}\} \ge 2))$ then
1.1) Return 1
2) Return 0

TestE (Input data: $j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, n_3, r, s, x, y, z, u$)
1) Let $m_3, m_4, d_3, d_4, t_3, t_4, q_3, q_4$ be defined as in (#)
2) If $j_1 = 0$ then
2.1) If not $m_3 = m_4 = 0$ then return 0
3) Else
3.1) If not $m_3, m_4 < 0$ then return 0
4) If $j_2 = 0$ then
4.1) If not $d_3 = d_4 = 0$ then return 0
5) Else
5.1) If not $d_3, d_4 < 0$ then return 0
6) If $j_3 = 0$ then
6.1) If not $t_3 = t_4 = 0$ then return 0
7) Else

7.1) If not $t_3, t_4 < 0$ then return 0
8) If $j_4 = 0$ then
8.1) If not $q_3 = q_4 = 0$ then return 0
9) Else
9.1) If not $q_3, q_4 < 0$ then return 0
10) If $j_3 = 2$ then
10.1) If $t_3 + t_4 \neq r$ then return 0
11) Else
11.1) If $t_3 + t_4 < r$ then return 0
12) If $d_3 + d_4 < s$ then return 0
13) Return TestF ($m_{22}, m_{23} - d_3, m_{24} - d_4, m_{33} - t_3, m_{34} - t_4 - q_3, m_{44} - q_4$)

TestD (Input data: $j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s$)
1) If $m_{12} < s$ then return 0
2) If $m_{11} \neq 0$ then return 0
3) If $m_{13} + m_{23} + 2m_{33} + m_{34} \neq 0 \pmod 3$ then return 0
4) If $m_{14} + m_{24} + m_{34} + 2m_{44} \neq 0 \pmod 4$ then return 0
5) $n_1 = m_{11} + m_{12} + m_{13} + m_{14}$
6) $n_3 = (m_{13} + m_{23} + 2m_{33} + m_{34})/3$
7) $n_4 = (m_{14} + m_{24} + m_{34} + 2m_{44})/4$
8) $n_1 \neq n_3 + 2n_4$ then return 0
9) If $m_{34} - n_3 + m_{13} + m_{23} = 0 \pmod 2$ then

9.1) If TestE $\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ m_{33}, (m_{34} - n_3 + m_{13} + m_{23})/2, m_{44}, m_{23} \end{pmatrix}$ then return 1

10) If $j_1 = 1$ then
10.1) If $m_{34} - n_3 + m_{13} + m_{23} = 0 \pmod 2$ then

10.1.1) If TestE $\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ m_{33}, (m_{34} - n_3 + m_{13} + m_{23})/2, 0, m_{23} \end{pmatrix}$ then return 1

10.2) If $m_{34} - n_3 + m_{13} + m_{12} - m_{24} = 0 \pmod 2$ then

10.2.1) If TestE $\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, \\ 0, (m_{34} - n_3 + m_{13} + m_{12} - m_{24})/2, m_{44}, m_{12} - m_{24} \end{pmatrix}$ then return 1

10.2.2) If TestE $\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, r-(m_{34}-n_3+m_{13}+m_{12}-m_{24})/2, \\ (m_{34}-n_3+m_{13}+m_{12}-m_{24})/2, m_{44}, m_{12}-m_{24} \end{pmatrix}$ then return 1

11) Else

11.1) If $(m_{34}-n_3+m_{13}+m_{23}=0(\bmod 2)$ and $m_{14}=0(\bmod 3))$ then

11.1.1) If TestE $\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, m_{33}, (m_{34}-n_3+m_{13}+m_{23})/2, \\ m_{14}/3-n_3+m_{13}-(m_{34}-n_3+m_{13}+m_{23})/2+m_{23}, m_{23} \end{pmatrix}$ then return 1

11.2) If $(m_{34}-n_3+m_{13}+m_{12}-m_{24}+m_{23}=0(\bmod 2)$ and $m_{13}=0(\bmod 2))$ then

11.2.1) If TestE $\begin{pmatrix} j_2, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, \\ m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s, m_{13}/2-(m_{34}-n_3+m_{13}+m_{12}-m_{24}+m_{23})/2, \\ (m_{34}-n_3+m_{13}+m_{12}-m_{24}+m_{23})/2, m_{44}, m_{12}-m_{24} \end{pmatrix}$

then return 1

12) If TestG $\begin{pmatrix} j_1, j_2, j_3, j_4, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s \end{pmatrix}$ then return 1

13) Return 0

TestC (Input data: $j_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, s$)

1) If $i_6=0$ then return TestD $\begin{pmatrix} i_1, j_2, i_4, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r, 0 \end{pmatrix}$

2) Elese if $((j_2=1)$ or $(m_{12}=0))$ then

3) For $i=0$ to $\min\left\{\left\lfloor \frac{m_{13}-2r}{2} \right\rfloor, m_{23}\right\}$

3.1) If TestD $\begin{pmatrix} i_1, 1, i_4, i_8, m_{11}, m_{12}+i, m_{13}-2\cdot i, m_{14}, \\ m_{22}, m_{23}-i, m_{24}, m_{33}, m_{34}, m_{44}, r, i \end{pmatrix}$ then return 1

4) Return 0

TestB (Input data: $i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r$)

1) If $(i_3=0)$ and $(i_5=0)$ then

1.1) return TestC $\begin{pmatrix} i_2, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$

2) Else if $(i_2=0)$ and $(i_3=1)$ and $(i_5=0)$

2.1) If $m_{22}\ge m_{12}$ then return $\begin{pmatrix} 1, i_1, i_4, i_6, i_8, m_{11}, m_{12}, m_{13}, m_{14}, \\ m_{22}-m_{12}, m_{23}, m_{24}, m_{33}, m_{34}, m_{44}, r \end{pmatrix}$

3) Else if $(i_2=1)$ and $(i_3=1)$ and $(i_5=0)$ then

3.1) If $(m_{12}\ge m_{22})$ then

3.1.1) If $\left(\gamma\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}=1\right)$ then return 1

3.2) Return $\left(\gamma\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}=1\right)$

4) Else if $(i_2=0)$ and $(i_3=0)$ and $(i_5=1)$ then

4.1) If $m_{22}\geq 2\cdot m_{12}$ then

4.1.1) If TestC $\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ m_{22}-2\cdot m_{12},m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}$ then return 1

5) Else if $(i_2=1)$ and $(i_3=0)$ and $(i_5=1)$

5.1) If $(2\cdot m_{12}\geq m_{22})$ and $(m_{22}=0(\bmod 2))$ then

5.1.1) If TestC$\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}$ then return 1

5.2) Return TestC$\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ m_{22}-m_{12},m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}$

6) Else if $(i_2=0)$ and $(i_3=1)$ and $(i_5=1)$ then

6.1) If $m_{22}\geq m_{12}$ then

6.1.1) If $2\cdot m_{12}\geq m_{22}$

6.1.1.1) If TestC$\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}$ then return 1

6.1.2) Return TestC$\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ m_{22}-m_{12},m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}$

7) If $(i_2=1)$ and $(i_3=1)$ and $(i_5=1)$ then

7.1) If $2\cdot m_{12}\geq m_{22}$ then

7.1.1) If TestC$\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ 0,m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}$ then return 1

7.2) Return TestC$\begin{pmatrix}1,i_1,i_4,i_6,i_8,m_{11},m_{12},m_{13},m_{14},\\ m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},r\end{pmatrix}$

8) Return 0

TestA (Input data: $i_1,i_2,i_3,i_4,i_5,i_6,,i_7,i_8,m_{11},m_{12},m_{13},m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44}$)

1) If $i_7=0$ then

1.1) If TestB$\begin{pmatrix}i_1,i_2,i_3,i_4,i_5,i_6,i_8,m_{11},m_{12},m_{13},\\ m_{14},m_{22},m_{23},m_{24},m_{33},m_{34},m_{44},0\end{pmatrix}$ then return 1

2) Else if $(i_7\neq 0 \text{ and } i_4=0)$ then

2.1) For $i=0$ to $\min\{m_{12},m_{13},m_{23}\}$

2.1.1) If TestB$\begin{pmatrix}i_1,i_2,i_3,2,i_5,i_6,i_8,m_{11},m_{12}-i,m_{13}+i,\\ m_{14},m_{22},m_{23}-i,m_{24},m_{33},m_{34},m_{44},i\end{pmatrix}$ then return 1

3) Else

3.1) For $i = 0$ to $\min\{m_{12}, m_{13}, m_{23}\}$

3.1.1) If TestB$\begin{pmatrix} i_1, i_2, i_3, i_4, i_5, i_6, i_8, m_{11}, m_{12} - i, m_{13} + i, \\ m_{14}, m_{22}, m_{23} - i, m_{24}, m_{33}, m_{34}, m_{44}, i \end{pmatrix}$ then return 1